UNIVERSIDAD AUTÓNOMA DE MADRID

FACULTAD DE CIENCIAS ECONÓMICAS Y EMPRESARIALES

DEPARTAMENTO DE ANÁLISIS ECONÓMICO: TEORÍA ECONÓMICA E HISTORIA ECONÓMICA

DOCTORAL DISSERTATION

The Impact of Social Security on the Stock of Physical Capital: New Approaches

By

MIGUEL SÁNCHEZ ROMERO

SEPTEMBER 2006

DIRECTOR: FELIPE SAEZ FERNÁNDEZ

A mi familia y, en especial, a mi abuelo

Antonio Romero Capilla

Contents

List of Figures

List of Tables

Agradecimientos

Esta tesis no hubiera sido posible sin la valiosa ayuda, apoyo y estímulo que durante los últimos años he recibido de muchas personas:

Por orden cronológico, al Profesor Barea, José Antonio Herce y Juan Francisco Jimeno por ser mi referente y modelo en el estudio de las pensiones.

Agradezco a José Manuel Rodriguez por darme la primera oportunidad de comenzar con lo que era y es mi vocación.

Maximino Cárpio y Paloma Tobes por confiar en mi profesionalidad para escribir el que ha sido mi primer artículo sobre pensiones. Del mismo modo, José Antonio Ortega por apoyar y avalar mi potencial para cumplir uno de mis sueños.

Gracias a todo el Departamento de Demografía de la Universidad de Califórnia, en Berkeley por su cálida acogida. En especial, Ronald Lee por aconsejar y guiarme en el camino correcto acerca del estudio del problema de la Seguridad Social. Alan J. Auerbach y Kenneth Wachter por sus comentarios y sus excelentes clases.

Al Departamento de Economía Cuantitativa de la UAM por su apoyo puntual a la hora de resolver algunos aspectos matemáticos. Muy en especial, debo agradecer en este sentido a Ester Gonzalez, Ana Primo y Eugenia Rosado.

El congreso internacional World Risk and Insurance Economic Congress (WREIC), en Utah, por los comentarios a un trabajo preliminar de la tesis. Principalmente a dos asistentes, Jason Seligman por sus sugerencias y a Jeffrey Brown, por proporcionarme artículos y darme ánimos cuando más lo necesitaba, reconociendo la valía del primer artículo de la tesis.

Gracias a la Universidad Autónoma de Madrid por la financiación recibida a través de la beca FPU, pues no sólo me ha permitido sobrevivir durante los últimos cuatro años, sino que también me ha aportado la estabilidad e independencia necesaria.

A mi Departamento, por aportarme el ambiente académico de tranquilidad y trabajo de los que he gozado durante el desarrollo de la tesis. Personalizo el agradecimiento en Fernando Belenguer, Maite Blazquez, Eva Camacho, Ainhoa Herrarte, Carlos Llano, Julián Moral, Richard Watt y José Luis Zofío por su apoyo y ánimo, así como sugerencias y comentarios, que han enriquecido cada uno de los capítulos de esta tesis.

Holly Fox y Richard Watt por sus consejos lingüísticos en inglés.

Mención especial merece la pena resaltar a mi director Felipe Sáez. Su confianza, estímulo y trabajo ha permitido terminar esta tesis, la cual nos parecía a priori inalcanzable.

Mis amigos y, en especial, Ester por todo el tiempo que se merecen y no les he podido dedicar.

Por último, a mi hermano y a mis padres les debo reconocer el inmenso esfuerzo para que dispusiera de las mejores condiciones, aunque para ello no haya estado físicamente a su lado en determinadas épocas. Mi abuelo, modelo de persona buena, inteligente, paciente y tolerante, que con su amor, silencios y excelentes consejos ha sido y será mi guía. Espero que esta tesis pueda ser motivo del orgullo que siempre me mostró.

Resumen

Garantizar un sistema público de pensiones, caracterizado por una población envejecida, es uno de los más importantes retos económicos que los países desarrollados deben hacer frente. En los últimos cuarenta años, las investigaciones acerca de las consecuencias que produce la Seguridad Social han sido muy abundantes, pero dos temas han centrado la mayor parte de las investigaciones: i) la pérdida de competitividad que origina la Seguridad Social, como consecuencia del efecto expulsión sobre el capital, y ii) la búsqueda de políticas económicas que contrarresten el futuro déficit del sistema de reparto provocado por el envejecimiento poblacional. En relación con esto último, algunas de las principales políticas sugeridas son: i) transformar el sistema de reparto a uno de capitalización, ii) desarrollar y promover la inversión en planes de pensiones privados, iii) modificar el cálculo de la pensión inicial y iv) retrasar la edad de jubilación, entre otros. Sin embargo, la gran mayoría de estas propuestas están basadas en modelos económicos de equilibrio parcial, en los cuales el agente económico no posee la capacidad de elegir entre planes de pensiones públicos y privados. Por consiguiente, el objetivo de esta tesis es: en primer lugar, analizar la conveniencia para el agente económico de un sistema público de pensiones. En segundo lugar, desarrollar un nuevo marco teórico con el fin de analizar no sólo los efectos a largo plazo derivados de un seguro público, sino también las consecuencias económicas que provoca la interacción de un seguro público y otro privado en la economía.

En el primer capítulo de la tesis, presentamos el modelo de elección óptima de consumo bajo incertidumbre vital de Yaari (1965). El objetivo que perseguimos en este capítulo es comprender porqué el agente económico no invierte la cantidad suficiente de sus recursos, para garantizar su pensión a través de seguros de vida y planes de pensiones privados, también denominado en la literatura económica como "annuity puzzle". Con este propósito, cuando no hay restricciones a la liquidez y los mercados son completos, demostramos que un individuo egoista puede no obtener una ganancia en su bienestar cuando asegura sus ahorros. Así, la decisión de contratar un seguro depende de la relación entre el valor presente de los ingresos futuros y el nivel de renta inicial. Este resultado extiende la teoría de la demanda de seguros y proporciona un marco idóneo para el estudio de los efectos económicos derivados de la existencia conjunta de seguros públicos y privados.

En el segundo capítulo introducimos una seguridad social de capitalización en el modelo de Yaari (1965). Con este nuevo marco teórico, analizamos los niveles de riqueza del agente económico, invertidos de manera coactiva por la Seguridad Social y de manera voluntaria en seguros de vida privados. Por medio de simulaciones obtenemos que una seguridad social de capitalización puede contribuir a alcanzar

mayores niveles de riqueza, incluso cuando el agente económico es egoista. Sin embargo, este resultado es muy sensible al tipo de cotización a la Seguridad Social y a los recargos sobre primas que impongan las compañías de seguros privados. Por ejemplo, un tipo de cotización a la Seguridad Social del 6 por ciento y mercados de seguros privados actuarialmente justos, incrementan la riqueza acumulada destinada a la pensión en un 17 por ciento, pero si los seguros privados no son actuarialmente justos, la riqueza acumulada puede decrecer en torno al 10 por ciento.

En el último capítulo desarrollamos un modelo de crecimiento económico con generaciones solapadas, en el cual los individuos son heterogéneos y desconocen, a priori, la edad a la que morirán. Con el doble objetivo de analizar el modelo de crecimiento económico con datos reales y no perder las propiedades que este modelo nos proporciona, el marco contable utilizado es longitudinal, en vez de transversal. Gracias a este cambio, demostramos que una economía clásica presenta estados estacionarios múltiples. La regla de oro y la regla de oro modificada son dos de los posibles equilibrios. Posteriormente, introducimos la Seguridad Social en el modelo para analizar qué efectos económicos produce esta institución en el largo plazo. Así, también demostramos que, bajo una población estable, tanto un sistema de reparto como un sistema de capitalización pueden alcanzar un crecimiento económico igual al crecimiento poblacional. Este último resultado implica que la seguridad social de reparto no produce nungún efecto expulsión a largo plazo del capital. Además, una seguridad social de reparto proporciona una mayor estabilidad a la economía que un sistema de capitalización.

Como síntesis del valor añadido de la tesis cabe destacar tres aspectos principales. Primero, la aportación realizada a la teoría de la demanda de seguros, gracias a la cual no sólo justificamos la escasa demanda de seguros de vida y planes de pensiones, sino también nos permite su incorporación dentro de los modelos de la teoría del consumidor. En segundo lugar, demostramos que la Seguridad Social puede mejorar, en el corto plazo, los niveles de riqueza de los agentes económicos, incluso cuando éstos son egoistas. Finalmente, extendemos la teoría del crecimiento económico con generaciones solapadas, por medio de la transformación del marco contable transversal en longitudinal. Esta transformación, por un lado, proporciona una mejora en el análisis de cualquier política económica, pues elimina la dificultad existente entre el uso de los modelos de crecimiento económico con generaciones solapadas y el análisis empírico de los efectos derivados del cambio poblacional y, por otro lado, proporciona un marco teórico-económico óptimo para el análisis conjunto de los seguros públicos y privados en una economía.

Chapter 1

Introduction

One of the most important economic challenges for developed countries is to guarantee a retirement income system in a society characterized by an aging population. In the future, this challenge will require not only an increasing flow of economic resources, but also a long-term agreement among politicians, firms, and labor unions. Exactly the same kind of long-term agreement took place more than a century ago, but under quite different socio-economic circumstances.

The introduction of the social security system by Prussian Chancellor Otto von Bismarck in 1889 helped members of the military to receive an income during their retirement, while at the same time recognizing their work for the country. Over the next 46 years more than twenty nations (mostly European) would develop some sort of social insurance. However, the first country to offer social security coverage to the entire society was the USA when the Social Security Act was passed in 1935. Both reforms were proposed under an economic situation of scarcity. Retirees, disables, widows, and orphans did not have enough wealth accumulated to maintain their economic status because industrialization had left aside those people who were unable to work. Hence, the economic situation coupled with a young population structure suggested that in the short run the most convenient social security was an unfunded system.

After six decades problems have arisen within the current demographic situation concerning the future feasibility of the unfunded social security system. For example, an aging population will excessively raise the fiscal pressure on future workers. This is because in an unfunded social security current workers pay the pension benefits of current retirees. Therefore, the unfunded system is based on a continuing intergenerational redistribution of economic resources, in which younger cohorts are worse off as the population ages.

We already know from the life-cycle theory of Modigliani and Brumberg (1954), and Ando and Modigliani (1957) that a payroll tax depresses personal savings. In addition to this, Feldstein (1974) pointed out that the pay-as-you-go system not only has a negative impact on personal savings, but it also distorts the labor supply by encouraging people to retire earlier, as well as by reducing its flexibility. As a consequence, the crowding out effect produced in the stock of physical capital could lead to an important reduction in the competitiveness of a country. Some of the solutions given to this problem include i) switching the unfunded system to a funded system, ii) developing annuity private markets, iii) finding formulae that reduce the pension benefits, or even iv) delaying the age of retirement, among other alternatives. In this thesis however, rather than enumerating other solutions, we directly analyze the root of social insurance. First, we will question whether or not a social security pension system is convenient, and in the case of being so, determine which social security system is the best. Second, we will develop a theoretical framework to analyze an economy with both public and private pension systems. This theoretical framework will evolve from a partial economic model with a representative agent into a growth economic model with heterogeneous people and realistic demography.

In general, the necessity of a social security has been politically justified by the myopic behavior of individuals.[1] This sort of behavior prevents people from saving the necessary amount of wealth for certain life circumstances such as retirement or unemployment. However, this justification is not generally accepted in economic theory. The economic literature, by contrast, has usually argued that a social secu-

[1]That is, because people are not rational, they are unable to correctly value their present and future resources. An experiment testing the assumptions of rational choice is Kotlikoff et al. (2001). They found that subjects displayed significant inconsistencies in their consumption decisions.

rity pension system is necessary due to the lack of a private annuity market. In fact, if people face an uncertain length of life and an annuity market does not exist, they will tend to anticipate future consumptions by depleting their wealth more quickly. In addition to the lifetime uncertainty, if financial institutions do not allow individuals to die in debt, then they will not save enough money for retirement. These conclusions are obtained from the model presented in the seminal paper of Yaari (1965). In this paper, he claimed that a selfish individual will fully annuitize her savings, so long as the annuity yield exceeds the risk-free yield and that is impossible for an individual to die in debt. Nevertheless, there is no empirical research that demonstrates this claim.[2] Hence, in the economic literature we find models that either assume an economy in which individuals fully annuitize their savings, or other models in which private annuities do not exist (thus, a social security system is needed, Eckstein et al. (1985), and Merton (1983), among others). Present-day developed countries usually have well developed private insurances as well as a social security pension system. However, people still do not voluntarily insure their wealth even with fiscal advantages. Therefore, in order to tackle the problem of the feasibility of the social security pension system, it is first necessary to develop a model in which both private and public pension systems exist and, for the sake of reality, in which individuals can voluntarily decide whether or not to purchase annuities. As a consequence, the first step is to solve the annuity puzzle.

During the past four decades economic research has tried to find out why individuals do not purchase annuities. Some of the reasons given for the annuity puzzle are: i) the desire to leave a bequest at the time of death, ii) the low return of private annuities, iii) the existence of close substitutes such as family transfers and social security benefits, and iv) imperfections in the annuity market such as the irreversibility of annuitization, and constant payouts. The first reason has been explored the most, however there does not exist a consensus with respect to the altruistic behavior, since by assuming a selfish individual who leaves an accidental

[2]This mismatch between the theoretical and empirical results has been called the "annuity puzzle".

or involuntary bequest,[3] we are almost able to obtain the stock of physical capital accumulated through generations, see Abel (1985) and Gokhale et al. (2001b). Second, even though U.S. government yields have exceeded annuity yields during many periods, there is other research, e.g. Friedman and Warshawsky (1990), that obtains opposite results without the expected increment in the demand for annuities. Third, Kotlikoff and Spivak (1981) showed that family transfers and social security benefits could affect the demand for private annuities by more than 70 percent. And yet Auerbach and Kotlikoff (1987) found that social security insurance does not significantly offset private life insurances. Consequently, to explain the limited demand for annuities, researchers usually assume that there exist market imperfections. More recently, Davidoff et al. (2005) have found that a large fraction of annuitized wealth remains optimal even with incomplete markets. They conclude that the lack of the demand for annuities is caused by behavioral biases. In this thesis, and more specifically in Chapters 2 and 3, we will develop a model that explain the small demand for private annuities following Davidoff et al. (2005)'s suggestion.

In Chapter 2, this model will help us to understand why individuals do not purchase annuities. And in Chapter 3, we will use this knowledge to examine the effects of social security on personal savings. Thus, we will be able to analyze, among other possibilities, the crowding out effect of social security on individual's wealth. The importance of this model with both public and private systems is twofold. First, alternative methods (besides the social security) for financing future consumptions at retirement must be found. *Ceteris paribus* the current economic situation, we already know that young cohorts will have to pay, due to the aging population, a greater payroll tax than current workers do. The second significance is that there currently is no model which is able to simultaneously analyze how the private annuity market and Social Security will evolve. In this sense, any economic policy that attemps to make the social security system feasible should take into account both systems.

Nevertheless, we should realize that the partial economic model of Chapters 2

[3]Wealth not consumed by a selfish individual at her death.

and 3 is not sufficient to analyze the effects of social security on economic growth. Economic variables such as wages, interest rates, and fiscal policies are continually modified. A dynamic perspective of the variables is also crucial in studying Social Security. And even more important seems to be the demographic transitions. Thus, in order to analyze this issue it is necessary to include an economic growth model.

The economic analysis of Social Security in a dynamic general equilibrium model is an accurate approach to the issue. However, it only allows an incomplete understanding, since it depends on simulations of particular cases. The growth model is for studying a more exact approach, for example, either the crowding out effect or the long-run welfare gains derived by a funded system relative to an unfunded system. Nevertheless, this latter model has an unresolved aggregation problem. People have different ages, different compositions of wealth, and different propensities to consume. Blanchard (1985) solved the aggregation problem by assuming that individuals face, throughout their life, a constant instantaneous probability of death. However, this assumption is the same as considering a representative agent, instead of heterogenous agents. Indeed, as Blanchard acknowledged, a constant instantaneous probability of death means that independently of the age of any individual, everyone have the same life expectancy, as well as propensity to consume. Calvo and Obstfeld (1988) considered a dynamic continuous model with realistic demography assumptions. Although, this paper is not prepared for empirical studies of the life-cycle using data bases.

In this thesis, the fourth chapter develops an overlapping generation (OLG) growth model, with realistic demography assumptions, in which we can examine the life cycle of any heterogeneous individual. A pioneer work in this field is Bommier and Lee (2003). Following Gale (1973), they developed an accounting framework in an OLG model that links cross sectional and longitudinal constraints. Converserly, we extend the Cass-Koopmans-Ramsey model by introducing a longitudinal accounting framework, instead of the cross-sectional account most frequently used.

This new longitudinal accounting framework, besides being more accurate for analyzing economic policies as it enables us to keep the track of each individual within the economy, leads us to a dynamic of consumption per capita not taken into account before. As a consequence, after comparing the funding system with the unfunded system, certain advantages of the funded system are undermined.

1.1 Organization of the thesis

The introduction is followed by three chapters. Chapter 2 presents a continuous life cycle model that will study the annuity puzzle. In order to gain insight into this puzzle, we will analyze consumption trajectories and utility levels. For the sake of simplicity we will focus on the lifetime period from 65 to 85 years old. The individual allocation process will be characterized by a behavioral bias suggested by Davidoff et al. (2005). Therefore, in addition to the dynamic optimization method, we will also use an algorithm based on the annuity equivalent wealth (AEW)[4] to determine whether the consumer decides to purchase annuities.

Chapter 3 deals with the effects of Social Security on the demand for private annuities. We will use and extend the model introduced in Chapter 2 in two different ways. First, we will apply the model to the life cycle, instead of doing so after the age of 65. Second, we will modify the utility function in order to introduce a portfolio with risky assets. Under this new framework, we will measure the impact of a funded social security on the accumulation of wealth by means of various simulations. Both public and private wealth profiles will be shown. Subsequently, we will determine those payroll taxes that could possibly increase individual wealth.

Chapter 4 presents an OLG growth model with realistic demography. This model uses a longitudinal accounting framework to analyze the life cycle of heterogenous group of people. Indeed, it is the theoretical framework not only to aggregate heterogenous groups with either perfect foresight or myopic behaviors, but also to introduce incomplete market conditions, such as liquidity constraints and private pension systems with a small demand for annuities. Nonetheless, in order to show the im-

[4]The proportion of annuitized wealth that is necessary to achieve the utility level when the consumer has no access to the annuity market.

portance of this new framework we will only examine the economy under "classic" conditions (perfect foresighted agents and complete Arrow-Debreu markets) and no social security. Later, we will study the effects of both social security systems on economic growth. In this chapter, we have used the dynamic optimization methodology, as in previous chapters, as well as the dynamic system theory.

Although the behavior of each individual within the economy is explained in each chapter, additional simulations contrasting the main results are included with tables and figures at the end of the chapter. Finally, Chapter 5 concludes the dissertation providing a summary of the thesis' findings and gives some lines of future research. Finally, we have only included the basic references cited through the dissertation. However, there exist a complementary bibliography that has not been included due to its extension.

Chapter 2

Welfare Gain and the Demand for Annuities

This chapter extends the annuity demand theory, giving new reasons for the small annuities demand. Regarding this problem, Yaari (1965) claims, under the condition that no one can die in debt, that a selfish consumer will fully annuitized her savings, insofar as annuity asset yield dominate conventional assets yield. However, we demonstrated mathematically that, in a standard life-cycle model, when borrowings are unconstrained and financial markets are complete, a selfish consumer may prefer not to annuitize her savings. In addition, we analyze the desire to purchase annuities according to the risk aversion coefficient and wealth composition.

2.1 Introduction

Upon retirement, individuals have to decide whether or not to finance their future consumption using annuities. Regarding this problem, conventional economic theory states that a selfish consumer will fully annuitized her savings, insofar as the yield of annuity assets dominates that of conventional assets. This result, obtained by Yaari (1965), was proved under the following conditions: i) the consumer's preferences are depicted by a constant relative risk aversion (CRRA) utility function, ii) the economic agent is a selfish consumer, iii) the yield of annuity assets dominates that of conventional assets, and iv) a negative asset position is forbidden when the consumer does not invest in annuities. Under these assumptions, however, there is no empirical research which supports Yaari's theorem. By contrast, many explanations have been proposed to give insights about the mismatch between the theoretical results and the small demand for annuities in the real world. For example, Bernheim (1991) suggests that consumers have altruistic feelings. Warshawsky (1988), Friedman and Warshawsky (1990) and Mitchell et al. (1999) report that the yield of private annuities does not exceed the market interest rate because of transaction costs. Kotlikoff and Spivak (1981) demonstrate that intra-family transfers may substitute annuities by more than 70 percent. Yagi and Nishigaki (1993) claim that it is optimal to allocate wealth not only in annuities, but also in conventional assets, when annuity payouts are constant. Even Brown and Warshawsky (2001) have pointed out other imperfections in the annuity market, such as the lack of protection against inflation, the irreversibility of annuitization and institutional barriers. However, research in the field is still fashionable, not only because the reduced demand for annuities continues to be puzzling for economists, but also because of the increasing concern in developed countries about the feasibility of the social security system.

Recently, Davidoff et al. (2005) have found that full annuitization is optimal under complete markets, under even less restrictive assumptions than those used by Yaari (1965), and also a large fraction of annuitized wealth remains optimal even with incomplete markets. Therefore, they conclude that the lack of the demand for annuities is caused by behavioral biases.

This chapter, following Davidoff et al. (2005), contributes to annuity demand theory demonstrating that a selfish consumer may prefer not to annuitize her savings, whenever a stream of future earnings is expected. This sort of behavior can be explained in two different ways. On the one hand, the result is obtained by weakening Yaari's fourth condition (see above) or, equivalently, by allowing the consumer to have a negative asset position upon death, but at longevity age.[1] On the other hand, an alternative explanation for the result is found by assuming an agent with shortsightedness, as is suggested by Davidoff et al. (2005). That is to say, the consumer is only worried about having positive wealth up to a certain date.[2]

Hence, we aim to give new insights on the small demand for annuities. To our knowledge, no one has mathematically proved that a selfish consumer under complete markets can be better off by not annuitizing her wealth. This unexpected result is only derived so long as there is a stream of future earnings. Thus, previous results under complete market settings are complementary to these, since researches[3] have thus far assumed an initial wealth, instead of a stream of earnings, which has to be allocated throughout the agent's lifespan.

The importance of this approach to the annuity puzzle is twofold. Firstly, it gives some theoretical clues concerning the macroeconomic consequences of private pension plans held by baby boomers, specially defined contribution (DC) plans. Secondly, it can give a better understanding of consumer behavior. The former is necessary for policy makers and insurance companies in order to encourage the demand for annuities; since the alternative asset depletes wealth faster, even to the point of outliving resources. Therefore, given the concern about the future feasibility of the Social Security in the developed countries, it is convenient to study different alternatives[4] which increase the demand for life annuity benefits by DC holders. This could contribute to a deeper knowledge on consumption-savings models. However, this point needs further research.

This chapter proceeds as follows: Section 2.2 reviews the most relevant literature about the demand for annuities. Section 2.3 describes the model that we use to determine optimal consumption and develops two subsections, which explain the reasons why the consumer may not decide to purchase annuities. The first subsection

[1] The final condition widely imposed on these models is zero wealth at longevity age.

[2] It is convenient to consider this date, in general, as her remaining expected life time, which is always less than the time necessary to reach longevity.

[3] This is not the case of Yaari (1965) and Hakansson (1969).

[4] Brown and Warshawsky (2001) suggest several alternatives concerning this point.

compares consumption trajectories under different investment possibilities. The second subsection is devoted to the anaylsis of the welfare gain from not purchasing annuities when a stream of future earnings is introduced into the model. Some tables containing the Annuity Equivalent Wealth (AEW) are presented. Section 2.4 concludes by pointing out the importance of the findings and suggesting directions for future research. The Appendix is located in Section 2.5 and Section 2.6 shows some simulations.

2.2 Overview of Life Annuities Demand

Annuity demand theory has experienced significant development since Yaari (1965)'s seminal paper, in which he proved that under the life-cycle hypothesis with an uncertain lifetime, the optimal behavior of a selfish individual is to hold all assets in the form of annuities. This statement is of extraordinary importance because, if confirmed, it supplies economists with an important tool that enables both a deeper knowledge of the consumer's behavior, and also it enables the implementation of more efficient public policies; e.g. do people save because of a concern for other people, or simply our of concern for themselves? Or, what is the best way to finance retirees income? Unfortunately, there is no empirical testing that confirms Yaari's result completely. In contrast, there seem to be several circumstances in which it becomes unattractive to purchase annuities.

The desire to leave a bequest at the time of death, according to the literature, is one of the most cited reasons for not fully annuitizing. However, and although Yaari (1965) already contemplates this alternative, there does not exist a consensus regarding the importance of the bequest motive on the consumer's behavior; and thus, whether or not the bequest motive can be used to justify the small demand for annuities. According to this literature, there are researchers, on the one hand, such as Abel (1985) or Gokhale et al. (2001a) who prove that accidental, or involuntary, bequests[5] are sufficiently rich to explain household inherited-wealth, reported by

[5] Wealth not consumed by a selfish individual at her death.

Kotlikoff and Summer (1981);[6] or Hurd (1989) in which it is determined that most bequests are accidental and desired bequests are small on average. Others like Brown (1999) point out that decisions about whether or not to annuitize DC plans are not affected by the bequest motive. But, on the other hand, Bernheim (1991) suggests that the existence of a strong bequest motive explains why people maintain a positive fraction of resources in bequeathable forms.

Another well known reason is the low return on private annuities. Of course it can not be guaranteed that a selfish consumer would prefer to purchase annuities when their yield does not exceed the market interest rate. Nevertheless, this is usually due to the price of fair annuities being increased by transaction costs, or loads,[7] which diminish the annuity yield below the risk free interest rate. For instance, Warshawsky (1988) shows for the United States that Government Bond yields exceeded annuity yields during the period 1919-1967. In contrast, Friedman and Warshawsky (1990) obtain the opposite result, using the same methodology, from 1968 to 1983. Finally, a more recent analysis, see Mitchell et al. (1999), has reported that the internal rates of return for SPIAs[8] available in 1995 were between 1 and 2 percent below the market returns.

In addition to these justifications, earlier researches have pointed out other imperfections in the annuity markets, as additional motives which decrease annuities demand. Brown and Warshawsky (2001)[9] suggest, among other possibilities, the following imperfections: i) the lack of protection against inflation, ii) the irreversibility of annuitization, and iii) institutional barriers and legal issues. Others have shown that constant payouts, offered by the private annuity system, is another motive for not fully annuitizing. This leads Yagi and Nishigaki (1993) to claim that it is optimal to allocate wealth both to annuities and to conventional assets, when annuity payouts are constant throughout the retirement period. Recently, Davidoff et al.

[6] They stated that roughly eighty percent of U.S. wealth was inherited.

[7] Annuity loads can be decomposed into the following three components: a) reserves, which are affected by the adverse selection problem, b) administrative costs and c) commissions and profits.

[8] Single-Premium Immediate Annuities (SPIA).

[9] Read this paper for a comprehensive extent literature review concerning these imperfections.

(2005) have proved, under less restrictive assumptions than those used by Yaari (1965), that positive annuitization remains optimal even with incomplete markets. As a consequence, they suggest either psychological or behavioral biases as possible explanations for the limited annuity demand.

Finally, private annuities have almost perfect substitute assets, such as family transfers and Social Security benefits, that obviously affect their demand. The former may substitute annuities by more than 70 percent, even when markets are complete. This is so, as was shown by Kotlikoff and Spivak (1981), because family members possess a level of information of each other that reduces problems like moral hazard, adverse selection and deception, which nevertheless do affect insurance companies. According to economic theory, insurance companies should substitute dollar for dollar for private annuities. Therefore, coupling this statement with the fact that Social Security benefits, in the developed countries, are the major flow of income for retirees, we can expect that private annuities should have a small demand. However, even with the existence of the social security system, Auerbach (1987) obtains the result that households do not significantly offset social security insurance by reducing their purchase of private life insurance. Consequently, we can conclude that, although several reasons have been suggested, the demand for annuities is still a puzzle for economists.

2.3 The Model

The aim of this section is to analyze consumption trajectories and utility levels under an uncertain lifetime. The analysis will help to understand both the psychological biases, and the economic circumstances, under which individuals are not willing to purchase annuities. Moreover, the chapter will show the main consequences derived from this irrational behavior.

We can assume under uncertain lifetime and complete markets that consumption can be financed, for the sake of simplicity, by two alternative assets; e.g. conventional assets and annuities. On the one hand, conventional assets yield a safe interest rate r

and have the important property that the investment is bequeathable. On the other hand, annuities are an actuarial contract between the consumer and an insurance company. This contract consists of paying, at the beginning of the period, a single premium to the insurance company in exchange for a lottery under which if the consumer survives at the end of the period, she will receive the safe interest rate r plus a risk premium μ contingent on her mortality risk; but, if she does not survive at the end of the period, she will not receive anything. Thus, while conventional assets are a bequeathable investment, annuities are not.

The consumption allocation process differs according to how important is the desire to leave bequests. However, we will follow the theoretical stream that assumes the consumer to be a selfish individual. Taking this into account, Yaari (1965) claims that the consumer will fully annuitize her savings. This theorem was proved under the condition of having a positive asset position throughout the lifespan. But, if the condition is relaxed, instead of financing consumption through annuities, the consumer may achieve a higher utility level by borrowing money and increasing present consumption. This result, although mathematically proved in the following pages, needs a further explanation.

The assumptions used by Yaari (1965) are rather close to reality. Therefore, it does not seem useful to weaken the borrowing constraint, unless we are suspicious that the consumer makes decisions subject to some irrationality. Following Davidoff et al. (2005), we propose that the economic agent behaves myopically and so, once the consumer has retired, she is only worried about having positive wealth during her remaining expected life, and not up to longevity. The main consequence derived from this behavior is a smaller demand for annuities, as well implying that consumers are more likely to outlive their resources.

We use the following additional necessary assumptions, in order to purchase actuarially fair life-annuities: i) the only source of uncertainty is the time of death. Longevity T is known in advance and is unalterable. ii) The probability of death Ω[10] is known and exogenous, and is also common to every consumer. iii) The mature

[10] See Definition 2.1 located in the appendix, page 28.

asset yields a known and safe interest rate. Nevertheless, and although these three conditions eliminate both the adverse selection problem (i.e. there are no unhealthier consumers who are more willing to purchase annuities than other healthier ones, at a given premium) and the ruin[11] problem, because the probability of death is known, there are two remaining loads which decrease the annuities internal rate of return. Consequently, if we assume that annuity yields at least dominate maturity yields, then we need to include that iv) there are no family agreements which permit transfers (see Kotlikoff and Spivak, 1981), and finally v) annuity markets are complete and their administrative costs are negligible.

Unfortunately, we have to realize that all of these conditions are not realistic. Nonetheless, they are necessary not only to analyze the myopic behavior, because of the possibility that the agent is not concerned about her future wealth as of a certain age; but also to calculate, on the one hand, the level of utility achieved by the consumer when her wealth is fully annuitized and, on the other hand, the level of utility achieved without purchasing annuities. The standard comparison between these two utility levels is labeled "Annuity Equivalent Wealth"[12] (AEW), and it will help later on to determine which asset is preferred by the consumer.

We shall proceed by introducing the mathematical problem faced by our consumer. The expected utility function U, which we assume to be similar for every consumer at a given age, will first be explained and, subsequently, two budget constraints will be presented. Each constraint will differ according to the asset selected to finance future consumption.

Firstly, the individual at age x, as a selfish consumer, is depicted by the following expected utility function, that she maximizes by the selection of a consumption plan,

$$U(x) = \int_x^T \frac{\Omega(s)}{\Omega(x)} \beta(s-x) u(c(s,x)) ds \text{ for all } x \in [0,T), \tag{2.1}$$

[11] The possibility that an insurance company will not have enough reserves, in order to satisfy the contracts, because of unexpected circumstances.

[12] See Brown (2003), Brown and Warshawsky (2001), Mitchell et al. (1999), Friedman and Warshawsky (1990), among others.

where in terms of Fisher-Yaari-Bommier[13] (FYB), we henceforth denote: $\frac{\Omega(s)}{\Omega(x)}$ as the (Fisher) rational discount function, $\beta(s-x)$ as the (Yaari) subjective discount function, and $\frac{\Omega(s)}{\Omega(x)}\beta(s-x)$ as the (Bommier) overall discount function. They measure the value at age x associated with a unit of consumption at age s. In particular, $\frac{\Omega(s)}{\Omega(x)}$ is the probability that an individual of age x will be alive at age s, while $\beta(s-x)$ is the widely used time discount factor from age x to age s, or $\mathrm{e}^{-\delta(s-x)}, \forall\delta \geq 0$. Hence, both $\Omega(\cdot)$ and $\beta(\cdot)$ are positive real functions valued less than or equal to one. On the other hand, $u(\cdot)$ is assumed to be a CRRA utility function of a risk averse consumer ($u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \gamma > 0$), like the one used by Yaari (1965). And, $c(s,x)$ represents the rate of expenditure[14] on consumption at age s, of an x year-old consumer.

This utility function has the important feature that, a single monetary unit of consumption delivers a greater utility as the consumer ages. This is due to property six of the rational discount function (see Definition 2.1, pag. 28). However, nowadays there exists a controversy about what type of rational discount function should be used. According to this, there are three main theoretical streams. The first supports the use of the common life tables, Hurd (1989), and Hurd and McGarry (1995). Another supports the necessity of continuously correcting the survival probability due to time misperceptions, specially close to young and old ages, Hamermesh (1985), Hurd and McGarry (2002) or Gan et al. (2003). And others like Bommier (2001, 2003a,b) state that more accurate results can be obtained using hyperbolic discounting, because individuals make inconsistent decisions. This chapter, however, will not enter into the discussion and so, we will plug the common life tables, as proxy values, into the rational discount function.

Secondly, the constraint faced by the consumer depends on whether she purchases annuities or not. Thus, we introduce two alternative budget constraints.

$$k(x) + \int_x^T \frac{R(s)}{R(x)}(w(s) - c(s,x))ds = 0, \tag{2.2}$$

[13] See Bommier (2001), Fisher (1977).

[14] We assume the rate of expenditure on consumption is a smooth function, $c \in \mathcal{C}^\infty([0,T) \times [0,T))$.

and

$$k(x)+\int_x^T \frac{R(s)}{R(x)}\frac{\Omega(s)}{\Omega(x)}(w(s)-c(s,x))ds=0. \tag{2.3}$$

(2.2) and (2.3) are respectively the budget constrained when consumption is financed by investing in conventional assets, and when consumption is financed by annuities. $k(x)$ is initial wealth at age x, and $w(s)$ is income at age s. There is no restriction about the source of income; so, $w(\cdot)$ may be either Social Security benefits when the agent is retired, or a salary if she continues working, or the sum of many different periodical earnings. On the other hand, $\frac{R(s)}{R(x)}$ is the financial present value at age x, of a monetary unit received at age s, and equivalently, $\frac{R(s)}{R(x)}\frac{\Omega(s)}{\Omega(x)}$ is the actuarial present value; that is,

$$\frac{R(s)}{R(x)}=\mathrm{e}^{-\int_x^s r(j)dj},$$

and

$$\frac{R(s)}{R(x)}\frac{\Omega(s)}{\Omega(x)}=\mathrm{e}^{-\int_x^s (r(j)+\mu(j))dj},$$

so that $r(j)$ is the safe interest rate at age j yielded by the conventional asset, and $r(j)+\mu(j)$ is the actuarially fair interest perceived in the case of being alive at the end of period j. Neither (2.2) and (2.3) constrain wealth to be nonnegative along the lifespan. Therefore, the consumer may be in debt at any time, although both (2.2) and (2.3) implicitly assume that wealth at longevity should be zero.[15]

2.3.1 *Consumption Trajectories*

In this model, our individual must decide whether to invest in annuities or not. Her decisions are subject to an irrational behavior known as myopia which, in this case, means that she is not worried about having a negative asset position from a certain date.[16] Moreover, for the sake of reality we assume that financial markets do not allow individuals to die in debt, and so we can expect that individuals will

[15]This is a necessary condition which represents the consumer's selfish behavior into the budget constraint.

[16]This statement is equivalent to the claim that the consumer assumes complete markets in which loans are allowed.

consume their income in each period. That is, we will follow Yaari (1965)'s problem only when the consumer reaches an asset position of zero. We suggest this behavior as a possible explanation for the small demand for annuities because it is usual to see how retirees, with Social Security benefits, prefer not to annuitize their DC plans, in order to enjoy their wealth at the beginning of their retirement, due to the higher probability of becoming unhealthy over the remaining expected lifetime. Nonetheless, the decision to purchase annuities depends on many more aspects which will be explained later on.

Firstly, according to this model the individual's welfare is based on her consumption trajectories. Therefore, we need to know not only how consumption increases over time, but also what the initial consumption is. This fact was studied previously by Barro and Friedman (1977), Davies (1981), Levhari and Mirman (1977), among others, who compare consumption trajectories when the consumer purchases annuities, with those trajectories when she does not. Optimal consumption growth, both in the case of investing in conventional assets, and in the case of purchasing annuities, are standard problems in optimal control theory. The results are:

$$\frac{\frac{\partial}{\partial s}c(s,x)}{c(s,x)} = \frac{r(s) - \mu(s) - \delta}{\gamma}, \tag{2.4}$$

and

$$\frac{\frac{\partial}{\partial s}\hat{c}(s,x)}{\hat{c}(s,x)} = \frac{r(s) - \delta}{\gamma}, \forall s \in [x, T). \tag{2.5}$$

Hereinafter, in order to distinguish the two alternative investments, we denote by $\hat{c}$ the consumption trajectory when annuities are purchased.

Thus, (2.4) represents the way in which consumption increases over time when an x year old consumer invests in conventional assets, while (2.5) does so when an x year old consumer invests in annuities. Note from (2.4) and (2.5) that consumption with an annuitized wealth grows faster. Concretely, over the lifespan, (2.4) decreases markedly when the consumer approaches longevity, while (2.5) leads to a continuously increasing consumption up to longevity, so long as $r(s) > \delta$ for all $s \in [x, T)$.

Secondly, if our aim is to understand why an individual at age x decides to allocate her wealth to conventional assets, instead of annuities, it is a necessary, but not sufficient, condition that at least her $c(x,x)$ should be greater than her $\hat{c}(x,x)$. According to this, we present in Table 2.1 the main factors that determine which initial consumption is greater:

Table 2.1: Determination of the Value $\frac{c(x,x)}{\hat{c}(x,x)}$

γ	$w(s) > 0$ †	$w(s) = 0$ ‡
$(0,1)$	> 1	> 1
$\{1\}$	> 1	$= 1$
$(1,\rightarrow)$	unknown result	< 1

† For at least one $s \in [x,T)$.

‡ For all $s \in [x,T)$.

To understand the meaning of Table 2.1 it is convenient to realize that the introduction of annuities in an uncertain lifetime model produces both a substitution effect and an income effect on initial consumption. This is so, because annuities offer a greater yield than conventional assets. In relation to this fact, Levhari and Mirman (1977) shows that when $w(s) = 0$ for all $s \in [x,T)$, life-span uncertainty affects optimal consumption in two opposing ways. On the one hand, the individual substitutes more future consumption for present consumption than under certainty. On the other hand, uncertain lifetime decreases (resp. increases) consumption because of having the possibility of a longer (resp. shorter) life. Moreover, the risk aversion coefficient γ together with a stream of future earnings plays a fundamental role in the determination of the value $\frac{c(x,x)}{\hat{c}(x,x)}, \forall x \in [0,T)$ (see Table 2.1). In particular, the risk aversion coefficient γ is necessary to calculate the present value of a future consumption. For example, whenever γ equals 1, a marginal unit of future consumption financed either by conventional assets, or by annuities, has the same marginal present value. Consequently, both initial consumptions will be alike. This fact shows up in column 3 (no future earnings) of Table 2.1. By contrast, assuming that $\gamma \in (0,1)$ (resp. $(1, \rightarrow)$), the future consumption financed by conventional assets is less (resp. more) preferred than the future consumption financed by annuities. Hence, because of the substitution effect $c(x,x)$ should be greater (resp. lower) than $\hat{c}(x,x)$. In the second term, the stream of future earnings also modifies the optimal initial consumption. Thus, the present value of a stream of future earnings under actuarially fair insurance is lower than under conventional assets. Then, future earnings have a negative effect on $\hat{c}(x,x)$ in comparison with $c(x,x)$.

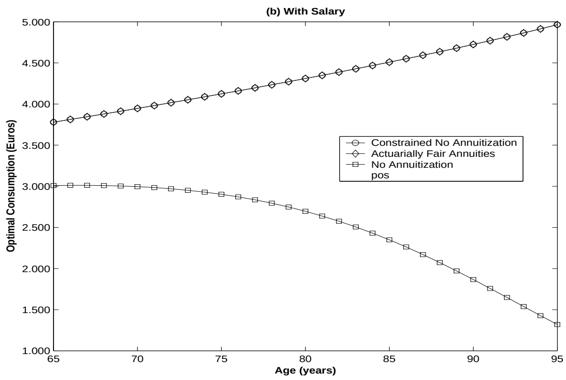
In short, when γ and the stream of future earnings are taken into account simultaneously, we can note (see columns 2 and 3) that $c(x,x)$ is greater than $\hat{c}(x,x)$ for a wider range of γ values. Concretely, the greater are future earnings with respect to initial capital $k(x)$, the wider is the range of γ values for which $c(x,x)$ is greater than $\hat{c}(x,x)$. So, assuming that γ belongs to the open range $(1, \rightarrow)$,[17] there exists a slight difference between considering an initial wealth, or a stream of future earnings. Nevertheless, this fact does not mean that the demand for annuities depends on the risk aversion coefficient. By contrast, we should take into account the fact that, when comparing $c(x,x)$ with $\hat{c}(x,x)$, there are two opposing effects. The less risk averse the consumer is, $\hat{c}(x,x)$ becomes lower with respect to $c(x,x)$ but, simultaneously, (2.5) becomes greater than (2.4), and vice versa. Therefore, the risk aversion coefficient leads to opposing results.

We find, however, that the difference between this result and those obtained previously, lies in the introduction of a stream of future earnings under complete markets. Therefore, it becomes interesting to analyze what the present value of future earnings is, with respect to initial wealth. For instance, in the appendix it is proved that, it is less (resp. more) likely that a retiree will decide to annuitize her DC plan when she has a much greater (resp. lower) accumulated capital in Social Security benefits. The implications of this statement are: i) individuals with resources in the form of capital (resp. non capital income) are more (resp. less) willing to

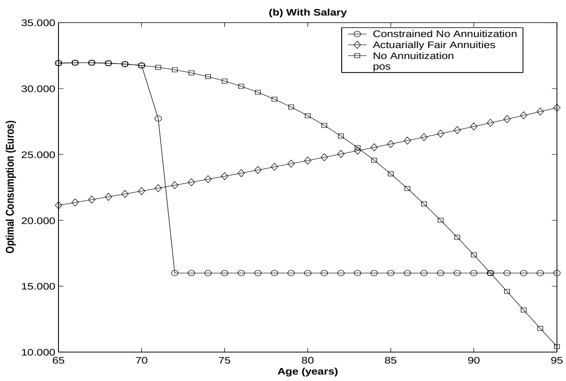
[17] Previous studies suggest that γ is typically above unity. For instance, Davies (1981) suggests with a similar model a value of 4 as the best guess.

purchase annuities; ii) Social Security benefits do not offset annuitized private pension plans completely, as Auerbach (1987) claims. In fact, there exists a threshold that is positively related with the present value of future earnings, above which the consumer will decide to annuitize her initial wealth. iii) individuals who choose not to purchase annuities outlive their initial wealth faster. Hence, their $c(x,x)$ will be greater than those individuals who have annuitized their wealth (or $\hat{c}(x,x)$), because they anticipate consumption. But they will consume exactly their income in each period from the date that they have depleted their wealth. A representation of such consumption trajectories is provided in Figure 2.1.

Figure 2.1: Selected consumption trajectories, $r = .02$, $\delta = 0$, $\gamma = 2$, $w = 16.000$, $k(65) = 100.000$



(a) Without Salary



(b) With Salary

These two cases represent the consumption trajectories when the only source of income is initial wealth (Figure 2.1.a) and, when besides the initial wealth, a periodic salary is introduced (Figure 2.1.b). The consumption trajectories are the cases shown in columns 2 and 3, row 3 of Table 2.1. Specifically, blue circles ($\circ$) plot the consumption path of a consumer who is able to borrow money and who does not purchase annuities. Red diamonds ($\diamondsuit$) plot the consumption of an individual who purchases actuarially fair annuities. And, black boxes ($\square$) plot the consumption trajectory when the myopic consumer does not purchase annuities and financial markets do not allow individuals to die in debt.

Note from Figure 2.1 that consumption under annuitized wealth, $\hat{c}(s,x)$ is not only always greater than $c(s,x)$ for all $s \in [x,T)$, but also it increases from year to year; while non-annuitized consumption decreases in Figure 2.1 from age 75. On the other hand, contrary to case (a), we are able to see from case (b) that non-annuitized initial consumption is greater than annuitized initial consumption. Therefore, see Figure 2.1.b, non-annuitized consumption is greater than optimal annuitized consumption up to age 76.

In conclusion, whenever γ belongs to the interval $(0,1)$, the early consumption financed by conventional assets is greater than the early consumption financed by annuities. Furthermore, $c(x,x)$ can exceed $\hat{c}(x,x)$ for γ-values that are greater than, although close to, 1 and the greater is the stream of future earnings. These facts, nevertheless, do not necessarily imply that specific values of γ lead to conventional assets being more preferred than annuities.

2.3.2 *Welfare Gains*

It is pointed out by Yaari (1965) and proved by Hakansson (1969) that a selfish consumer will fully annuitized her savings, so long as she cannot die in debt. Subsequently, Kotlikoff and Summer (1981), Hurd (1989), Mitchell et al. (1999) and Brown and Warshawsky (2001) show that a selfish consumer will fully annuitize her initial wealth, even with complete markets. Recently, Davidoff et al. (2005) has extended the annuity demand theory by presenting less restrictive conditions under complete markets, under which we still get full annuitization. However, it has not been proved mathematically that a selfish consumer may prefer not to annuitize her savings under complete markets, when both an initial wealth, and a stream of future earnings are taken into account simultaneously.

We shall proceed by explaining the circumstances under which individuals are not willing to purchase annuities. The explanation follows two steps. Firstly, we show, with the help of Table 2.2 below, that the risk aversion coefficient does not explain the demand for annuities, at least for the most common range of γ-values used by economists. Secondly, we state Proposition 2.1, which is the main finding of the chapter, and we shall outline its consequences upon the consumption allocation process.

We assume that a consumer decides to purchase annuities, so long as this asset leads to a welfare gain with respect to the alternative investment. In general, a common and easy way to prove that a welfare gain exists is by means of the annuity equivalent wealth (AEW), which is the proportion of annuitized wealth, that is necessary to achieve the maximum utility level when the consumer has no access to the annuity market. In other words, AEW offers an intuition about whether or not the indifference curve under annuitized wealth is greater. Concretely, if this proportion is greater (resp. less) than one, annuities (resp. conventional assets) cause a welfare gain to the consumer. Table 2.2 reports the AEW values associated with the problem depicted in Figure 2.1, when γ equals 2. Nevertheless, we have introduced two additional cases in order to analyze the role of the risk aversion coefficient.

Table 2.2: Annuity Equivalent Wealth (AEW) Values

Age	$w(s) > 0^{\dagger}$			$w(s) = 0^{\ddagger}$		
	$\gamma = .75$	$\gamma = 2$	$\gamma = 5$	$\gamma = .75$	$\gamma = 2$	$\gamma = 5$
65	0.8982	0.9653	1.0389	1.4302	1.5342	1.6472
70	0.8011	0.9151	1.0440	1.4963	1.6412	1.7912
75	0.6993	0.8165	1.0492	1.5743	1.7740	1.9751
80	0.6472	0.7535	1.0547	1.6644	1.9369	2.2070
85	0.5940	0.7181	1.0612	1.7643	2.1317	2.4925
90	0.5450	0.6871	1.0719	1.8692	2.3540	2.8257
95	0.5107	0.6712	1.0943	1.9708	2.5860	3.1626

† For at least one $s \in [x, T)$, and ‡ for all $s \in [x, T)$.

Note: In this table, the mortality hazard rate is assumed to follow the Gompertz's Law ($\mu(x) = \alpha e^{\lambda x}$), where α is equal to $exp(-11.74311)$ and $\lambda = 0.106402$, for all x between 65 and 110 years old. Further, it has been used the same values from Figure 2.1, in order to illustrate which consumption yields a greater utility in each case.

The values shown in each column correspond to the AEW of our myopic consumer, who starts at the age of 65 with an initial wealth of 100.000 euros and, for the case of receiving a periodical endowment, we assume an annual salary of 16.000 euros. Moreover, the values reported in Table 2.2 take into account the wealth already consumed. That is, the consumer's wealth decreases from year to year. This is the reason why the individual reinforces her decision to either finance the consumption through annuities, or to finance her consumption through conventional assets. So, once the consumer has invested in a specific asset, we may expect that she will hold the investment.

Now, if we pay attention to columns 3 and 6 of Table 2.2, we can see the AEW associated with Figure 2.1 (specifically, consumption trajectories plotted with black boxes). Table 2.2 (column 6) confirms that the consumer achieves a higher utility level by annuitizing her wealth, see Figure 2.1.a; while, in the case of Figure 2.1.b, the consumer prefers to increase her early consumption by investing in conventional assets, see Table 2.2 (column 3). However, the latter consumption trajectory depletes wealth faster and, as a consequence, the individual only can consume her income as of age 77.

It has been pointed out that the decision as to whether to annuitize or not does not depend on the risk aversion coefficient, Levhari and Mirman (1977). This result also shows up in Table 2.2. Note that the AEW values do not change drastically when the consumer's risk aversion, at the age of 65, varies from 0.75 to 2. In fact, given the resources assumed for the consumer, conventional assets yield greater welfare than annuities for γ values up to 3 (whenever there is a stream of future endowments). Thereinafter, the consumer will prefer to annuitize her wealth; e.g. $\gamma = 5$ as is shown in Table 2.2. Hence, the risk aversion coefficient does not explain the demand for annuities, because there exists a γ value, for which the consumer changes the investment in conventional assets to annuities, that i) depends on Proposition 2.1 and ii) its value is, in general, outside the common range of γ values obtained by economists.[18]

Proposition 2.1 *When the individual is selfish and faces an uncertain lifespan, the decision of whether or not to purchase an asset contingent upon her death depends on the relationship between the present value of future earnings and the initial wealth.*

Proposition 2.1 is principally of interest for understanding the small demand for annuities. Being specific, annuities are less (resp. more) preferred than conventional assets, the greater (resp. lower) is the present value of future earnings with respect to the initial wealth. Note that we are considering a selfish individual. So, it is not a necessary condition that the consumer has altruistic feelings, or a bequests motive, in order not to purchase annuities. Furthermore, Proposition 2.1 assumes that not only does the yield of annuities dominate that of conventional assets, but also annuities are actuarially fair. Therefore, we can extract from Table 2.2 the following behavior: i) our myopic consumer may not purchase actuarially fair annuities, when there is a stream of future earnings; but ii) when the consumer is willing to purchase assets contingent on her death, the insurance company can supply unfair annuities. Table 2.3 below reports the maximum loading that the insurer can charge to the

[18]The mathematical proof is found in the Appendix, page 28.

consumer. Finally, iii) when initial wealth is the only source of income, we find that the consumer purchases annuities, even with unfair premiums (although the yield of annuities must dominate that of conventional assets).

Table 2.3: Maximum Loads† (ϖ)

$k(65)$	ϖ		
	$\gamma = .75$	$\gamma = 2$	$\gamma = 5$
100.000	*	*	.4000
150.000	*	.0950	.6850
200.000	*	.3150	.8500
250.000	.0550	.4825	.9600

* Conventional assets yield a greater welfare.

Note: $w(s) = 16.000$ for all $s \in [x, T)$. The formula used to calculate the maximum load is as follows: $r(s) + (1 - \varpi) \cdot \mu(s)$.

Table 2.3 illustrates how a consumer demands a greater yield, the lower is the initial wealth with respect to the present value of future earnings (look at each column from the bottom to the top). Another important finding shows up when the consumer does not have earnings during future periods. In this particular case, our individual usually decides to annuitize.

In short, whenever DC-plan holders behave as our myopic consumer, we can state that Social Security benefits, or existing annuities, produce a crowding out effect on brand new annuities. However, this crowding out is not always dollar for dollar. In particular, the decision to annuitize new wealth depends on both future income, and on psychological aspects (e.g. risk aversion and subjective time discount). Hence, DC-plan holders with high Social Security benefits, or existing annuity payouts, will not annuitize their plans, unless the capital compounded in their DC pension plans is thought to be high enough. So, they choose a financial payout, which allows the possibility of leaving bequests.

2.4 Conclusions

This chapter contributes to the theory of the demand for annuities, providing new insights on the annuity puzzle. We show that annuities may not cause a welfare gain relative to conventional assets. Consequently, the consumer is not willing to purchase annuities and so, she is more exposed to the risk of outliving her resources at the end of her life. This situation can be explained in two different ways. On the one hand, by relaxing the borrowing constraint condition imposed in Yaari (1965). On the other hand, by following Davidoff et al. (2005), and assuming that the economic agent is only concerned with having positive wealth during her remaining expected life, instead of doing so up to longevity. So, once one of these approaches is taken into account, we have found, firstly, that the risk aversion coefficient does not explain the demand for annuities and, secondly, that the decision as to whether or not to purchase an asset that is contingent upon her death, when the individual is selfish and faces an uncertain lifespan, depends on the relationship between the present value of future earnings and initial wealth.

The theoretical finding has important implications for the social security system, for private pension plans and for insurance companies. On the one hand, for example, Social Security does not offset, at the age of retirement, a dollar in public annuities by reducing a dollar in the private market. In particular, the crowding out effect is lower the greater is the wealth that is accumulated in the private market. On the other hand, we have shown how those economic agents, who finance their consumption through annuities, are willing to purchase actuarially unfair premiums, even to the point of demanding annuities with half of the risk premium contingent upon her death.

Finally, the conclusions obtained in this chapter suggest extentions of the analysis, to include different annuities, such as SPIAs or constant annuity payouts, as well as to study the effects that fiscal policies can produce upon the demand for annuities.

2.5 Appendix

In this appendix we shall present the main properties of the rational discount function or mortality hazard rate. Each property is essential both to determine the optimal allocation and to calculate the actuarial fair annuity. Subsequently, the proof of proposition 2.1 will be sketched.

Definition 2.1 *Let $\Omega \in \mathcal{C}^2([0,T))$ denote the rational discount function, which means the probability that the consumer will be at age x. Ω has the following properties:*

1. $\Omega(0) = 1$.

2. $\lim_{x \to T} \Omega(x) = 0$.

3. $0 < \Omega(x) \leq 1$.

4. $\Omega(s) < \Omega(x) \Leftrightarrow s > x$.

5. $-\frac{\frac{\partial}{\partial x}\Omega(x)}{\Omega(x)} = \mu(x) > 0$.
   where $\mu \in \mathcal{C}^\infty([0,T))$ is the instantaneous mortality rate, also known by demographers as "mortality hazard rate", or by actuaries as "force of mortality".

6. *The mortality hazard rate μ is an increasing function on age*[19]
   $\frac{\partial}{\partial x}\mu(x) \geq 0$, $\forall x \in [0,T)$.

7. *In particular, the probability of being alive at age x is given by the following mapping:*

$$\begin{array}{rcl} \Omega : [0,T) & \rightarrow & (0,1] \\ x & \mapsto & \Omega(x) = \mathrm{e}^{-\int_0^x \mu(s)ds} \end{array}$$

Proof Proposition 2.1. Let there be two alternative indirect utility functions which are labeled V and W respectively. The former depicts the utility derived by

[19]In general, this is not true at young ages, although it is assumed for the sake of simplicity.

investing in actuarially fair annuities, while the latter does investing in conventional assets. Each indirect utility function has the following mapping:

$$V(x) = \frac{Ta(x)^{1-\gamma}}{1-\gamma} \left(\int_x^T \frac{\Omega(s)}{\Omega(x)} \left(\frac{R(s)}{R(x)} \right)^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds \right)^{\gamma}, \tag{2.6}$$

and

$$W(x) = \frac{Tc(x)^{1-\gamma}}{1-\gamma} \left(\int_x^T \left(\frac{\Omega(s)}{\Omega(x)} \right)^{\frac{1}{\gamma}} \left(\frac{R(s)}{R(x)} \right)^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds \right)^{\gamma}. \tag{2.7}$$

Where $Ta(x)$ is equal to the initial wealth plus the present value of future earnings discounted by actuarially fair annuities,

$$k(x) + \int_x^T \frac{\Omega(s)}{\Omega(x)} \frac{R(s)}{R(x)} w(s) ds \geq 0,$$

and, similarly, $Tc(x)$ is equal to the sum of the initial wealth and the present value, at age x, of the same future earnings discounted by conventional assets

$$k(x) + \int_x^T \frac{R(s)}{R(x)} w(s) ds \geq 0.$$

Note that $Ta(x) \leq Tc(x)$ for all $x \in [0, T)$, because of Definition 2.1.

We aim to obtain whether or not (2.7) can be greater than (2.6). To do so, we shall proceed dividing (2.6) over (2.7) in order to simplify the algebra.

$$\frac{V(x)}{W(x)} = A(x)^{1-\gamma} \cdot B(x, \gamma)^{\gamma}, \tag{2.8}$$

where

$$0 < A(x) = \frac{Ta(x)}{Tc(x)} \leq 1,$$

and

$$B(x, \gamma) = \frac{\int_x^T \frac{\Omega(s)}{\Omega(x)} (\frac{R(s)}{R(x)})^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds}{\int_x^T (\frac{\Omega(s)}{\Omega(x)})^{\frac{1}{\gamma}} (\frac{R(s)}{R(x)})^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds} > 0.$$

Therefore, if (2.8) is greater (resp. less) than the unit, (2.6) is more (resp. less) preferred than (2.7). This result depends mainly on function $B(x, \gamma)$ due to $A(x)$ is always less or equal than one. According to this fact, we shall study the image of the function $B(x, \gamma)$ firstly

$$B(x,\gamma)\begin{cases} >1 & \text{if} \quad 0<\gamma<1 \\ =1 & \text{if} \quad \gamma=1 \\ <1 & \text{if} \quad \gamma>1 \end{cases}, \forall x \in [0,T).$$

In order to demonstrate that $B(x,\gamma)$ is greater than one, for all $0<\gamma<1$, we study the values of the function $\Omega(s)^{\frac{1}{\gamma}}$ relative to $\Omega(s)$.Thus, let $\gamma>0$, such that $\gamma<1$, then $\frac{1}{\gamma}>1$. If $\Omega(s)$,$\forall s \in [0,T)$, is a positive real function lower than one, we have that $\Omega(s)^{\frac{1}{\gamma}}<\Omega(s)$. Now, multiplying and integrating both sides of the inequality by the same positive function, the inequality holds that

$$\int_x^T \left(\frac{\Omega(s)}{\Omega(x)}\right)^{\frac{1}{\gamma}} \left(\frac{R(s)}{R(x)}\right)^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds < \int_x^T \frac{\Omega(s)}{\Omega(x)} \left(\frac{R(s)}{R(x)}\right)^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds.$$

To obtain the inequality sense of the function $B(x,\gamma)$ let divide the right side of the inequality by the left side, it results that

$$B(x,\gamma)=\frac{\int_x^T \frac{\Omega(s)}{\Omega(x)} \left(\frac{R(s)}{R(x)}\right)^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds}{\int_x^T \left(\frac{\Omega(s)}{\Omega(x)}\right)^{\frac{1}{\gamma}} \left(\frac{R(s)}{R(x)}\right)^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds} > 1, \forall\gamma \in (0,1), \text{ and all } x \in [0,T).$$

On the other hand, when $\gamma>1 \Rightarrow 0<\frac{1}{\gamma}<1$, which implies that $\Omega(s)^{\frac{1}{\gamma}}>\Omega(s)$. Using the preceding reasoning, we obtain for the range of γ values that

$$B(x,\gamma)=\frac{\int_x^T \frac{\Omega(s)}{\Omega(x)} \left(\frac{R(s)}{R(x)}\right)^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds}{\int_x^T \left(\frac{\Omega(s)}{\Omega(x)}\right)^{\frac{1}{\gamma}} \left(\frac{R(s)}{R(x)}\right)^{1-\frac{1}{\gamma}} \beta(s-x)^{\frac{1}{\gamma}} ds} < 1, \forall\gamma > 1, \text{ and all } x \in [0,T).$$

Finally, when $\gamma=1$, the functions $\Omega(s)^{\frac{1}{\gamma}}$ and $\Omega(s)$ are equal, therefore is easy to prove that $B(x,\gamma)=1$.

So far, we have studied the image of $B(x,\gamma)$ because of its influence over (2.8). However, we can simplify the algebra by taking logarithms on (2.8), and so, we can achieve the solution by analyzing if the transformation is either positive or negative.

$$\ln\left(\frac{V(x)}{W(x)}\right)=(1-\gamma)\cdot\ln(A(x))+\gamma\cdot\ln(B(x,\gamma)). \tag{2.9}$$

To determine the signal of (2.9), let split the set of γ-values into three parts, such that

1. When $\gamma \in (0,1)$, the $\ln(B(x,\gamma))$ is always positive. Therefore, (2.9) takes the following values

$$\ln\left(\frac{V(x)}{W(x)}\right)\begin{cases} >0 & \Leftrightarrow \quad 1>\gamma>\frac{1}{1-\frac{\ln(B(x,\gamma))}{\ln(A(x))}}, \\ <0 & \Leftrightarrow \quad 0<\gamma<\frac{1}{1-\frac{\ln(B(x,\gamma))}{\ln(A(x))}}<1. \end{cases} \tag{2.10}$$

2. When $\gamma = 1$, the sense of the inequality between $V(x)$ and $W(x)$ is undetermined. Nonetheless, $V(x) > W(x)$ whenever $Ta(x) = Tc(x)$.

3. Finally, when $\gamma > 1$, (2.9) gives

$$\ln\left(\frac{V(x)}{W(x)}\right)\begin{cases} <0 & \Leftrightarrow \quad \gamma>\frac{1}{1-\frac{\ln(B(x,\gamma))}{\ln(A(x))}} \text{ and } \ln(B(x,\gamma)>\ln(A(x)), \\ >0 & \qquad \text{Otherwise.} \end{cases} \tag{2.11}$$

$\blacksquare$

2.6 Simulations

The following tables and figures show how under different scenarios our individual consumes her wealth from age 65 to age 85. These data, obtained through various simulations, help us to understand the main factors that determine whether an individual prefers to purchase annuities or not. Thus, we will demonstrate that:

> *When the individual is selfish and faces an uncertain lifespan, the decision of whether or not to purchase an asset contingent upon her death depends on the relationship between the present value of future earnings and the initial wealth.*

We have divided this subsection into two cases: men and women. We have also considered two initial wealths {150.000, 300.000}. The simulations have been obtained by assuming that: i) the population grows at a constant rate equal to .015, ii) every individual discounts future consumptions both subjectively and rationally, iii) the subjective discount factor is constant over time and equal to .01, iv) the rational discount is a survival probability function. Regarding this last assumption, we have introduced some demographic tables, which contain the mortality hazard rates, for four Spanish cohorts (Tables from 2.4 to 2.7).[20] In particular, we have calculated the mortality hazard rates of the Spanish cohorts born in the years 1940, 1960, 1980, and 2000. The underlying reason for considering four different cohorts is to analyze the effects of an increment in the life expectancy on our individual's allocation process. On the other hand, in order to analyze the effects of the risk aversion coefficient on the demand for annuities, we have considered three feasible values of γ: .75, 2, and 5. Finally, based on Chapter 4 we have assumed that vi) the economy can achieve either the *modified golden rule* or the *golden rule* steady-state. As a consequence, we simulate the allocation processes under four different stationary interest rates

[20]In order to estimate this data we have used the following data base:
Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on [2004]).

(and wages) {.015, .02125, .04, and .085}. Later on, we will use these simulations to compare the different allocation processes obtained in chapters 3 and 4.

According to the assumptions, we are able to note looking at both tables (2.13 to 2.21) and figures (2.3 to 2.50) that the decision to purchase annuities at the age of retirement (here 65 years old) depends, not on the survival probability and the risk aversion, but rather on the relationship between the pension benefit b, paid by an unfunded Social Security, and the initial wealth $k(65)$. In particular, if wealth decreases sharply or is equal to zero, then the individual will not have purchased annuities. In contrast, if wealth at 85 years old is positive, it will mean that our individual will have purchased annuities. Consumption trajectories, on the other hand, show the consequences of not purchasing annuities. For example, a consumption trajectory with a steep downward-slope means that the individual has not purchased annuities. Her consumption therefore rises at the beginning of her retirement period, but it decreases sharply afterwards, up to the amount received in pension benefits. A smooth consumption from age 65 to 85, by contrast, depicts an individual who has purchased annuities. Finally, a consumption trajectory that increases rapidly and, subsequently, decreases represents an individual who has purchased annuities at the beginning of her retirement period, but has changed her decision afterwards.

In sum, we see that, on the one side, even though women and men have different survival probabilities (look at figure 2.2), and so life expectancies, their decision of purchasing annuities still depend on the relationship between the pension benefit and the initial wealth. In order to compare the consumption trajectories of both genders, we need to choose a table in which both the pension benefit and the initial wealth will be similar. For example, tables 2.14 and 2.18 satisfy this condition. Thus, if we compare figures from 2.9 to 2.14 with those from 2.27 to 2.32, we will see that consumption trajectories are similar. On the other side, besides the lack of influence of the survival probability upon the demand for annuities, we also realize from the figures and tables below that the risk aversion coefficient is not the main factor to analyze. Concretely, the risk aversion only postpones current consumption to a future date. However, it does not modify the decision of purchasing annuities;

for examples, see figures from 2.3 to 2.5.

Table 2.4: Survival Curves: Men

Age	1940 l_x	1940 e_x	1960 l_x	1960 e_x	1980 l_x	1980 e_x	2000 l_x	2000 e_x
65	28779	10.53	69382	12.92	76905	14.61	81932	16.77
70	20175	8.01	58518	9.95	67075	11.35	74234	13.24
75	11649	6.09	44377	7.46	53532	8.61	63055	10.12
80	5020	4.67	28486	5.52	36966	6.35	48101	7.45
85	1558	3.48	14113	4.05	20263	4.64	30308	5.35
90	277	2.90	4824	3.16	7813	3.44	13968	3.81
95	39	2.13	1148	2.29	1916	2.60	4080	2.75
100	2	1.74	141	1.82	260	2.01	645	2.05
105	0	1.49	9	1.51	18	1.63	49	1.62
110	0	1.34	0	1.34	1	1.42	2	1.39

l_x the number of surviving members (from a sample of 100000 people) of the cohort at age x.

e_x expectation of future life beyond age x.

Table 2.5: Mortality Hazard Rates (Men): $\mu(x) = \alpha e^{\beta x}$

Year	$log(\alpha)$ Mean	$log(\alpha)$ (Std. Dev.)	β Mean	β (Std. Dev.)
1940	-7.464593	(.247149)	.068561	(.002916)
1960	-8.693344	(.145519)	.080501	(.001696)
1980	-9.291359	(.114363)	.085277	(.001324)
2000	-10.131433	(.101554)	.092800	(.001169)

Table 2.6: SURVIVAL CURVES: WOMEN

Age	1940		1960		1980		2000	
	l_x	e_x	l_x	e_x	l_x	e_x	l_x	e_x
65	50973	12.66	78754	15.21	87957	17.86	92215	20.54
70	42726	9.63	70289	11.72	82095	13.94	88395	16.31
75	30680	7.46	57190	8.80	72295	10.46	81915	12.39
80	18838	5.69	39768	6.52	56928	7.57	70594	8.94
85	9201	4.32	22012	4.80	36295	5.42	51836	6.22
90	3220	3.41	8727	3.64	16994	3.88	28511	4.24
95	890	2.69	2430	2.65	5126	2.82	9793	2.99
100	136	2.19	355	2.05	854	2.10	1831	2.12
105	13	1.83	27	1.66	69	1.65	150	1.61
110	1	1.62	1	1.44	3	1.41	5	1.36

l_x the number of surviving members (from a sample of 100000 people) of the cohort at age x.

e_x expectation of future life beyond age x.

Table 2.7: MORTALITY HAZARD RATES (WOMEN): $\mu(x) = \alpha e^{\beta x}$

Year	$log(\alpha)$		β	
	Mean	(Std. Dev.)	Mean	(Std. Dev.)
1940	-8.136371	(.285225)	.073050	(.003344)
1960	-9.663813	(.167940)	.088782	(.001945)
1980	-11.121971	(.164062)	.102784	(.001888)
2000	-12.529926	(.175215)	.115683	(.001992)

Figure 2.2: Survival Curves for Different Spanish Cohorts

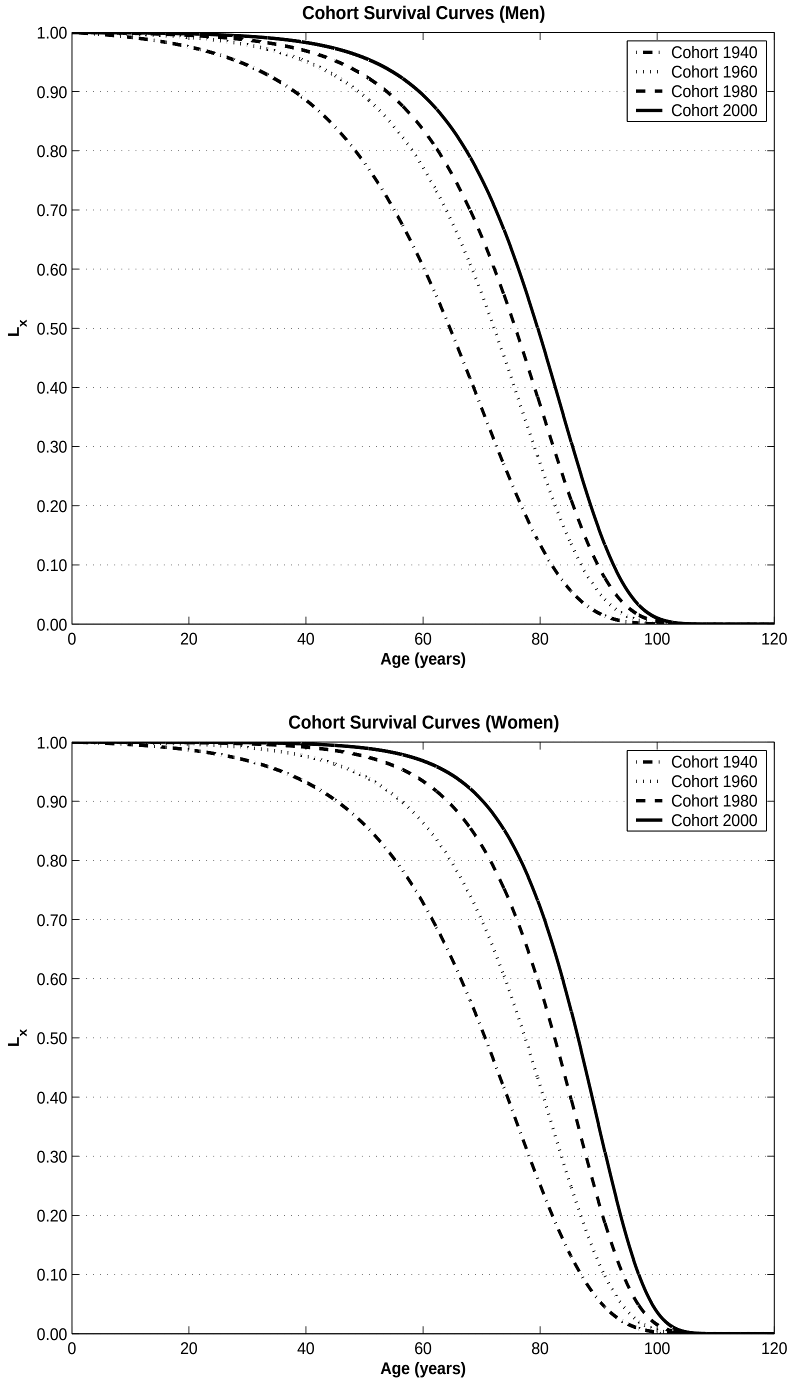


Table 2.8: ANNUITY YIELDS FOR THE COHORTS (MEN): 1940 AND 1960

Age	Mature Assets	1940 Fair	1940 $\varpi = .25$	1940 $\varpi = .50$	1940 $\varpi = .75$	1960 Fair	1960 $\varpi = .25$	1960 $\varpi = .50$	1960 $\varpi = .75$
65	.0150	.0627	.0508	.0389	.0269	.0452	.0376	.0301	.0225
66	.0150	.0661	.0533	.0406	.0278	.0477	.0395	.0314	.0232
67	.0150	.0697	.0561	.0424	.0287	.0504	.0416	.0327	.0239
68	.0150	.0736	.0590	.0443	.0297	.0534	.0438	.0342	.0246
69	.0150	.0778	.0621	.0464	.0307	.0566	.0462	.0358	.0254
70	.0150	.0822	.0654	.0486	.0318	.0601	.0488	.0376	.0263
71	.0150	.0870	.0690	.0510	.0330	.0639	.0517	.0395	.0272
72	.0150	.0921	.0728	.0536	.0343	.0680	.0548	.0415	.0283
73	.0150	.0976	.0770	.0563	.0357	.0725	.0581	.0437	.0294
74	.0150	.1035	.0813	.0592	.0371	.0773	.0617	.0461	.0306
75	.0150	.1097	.0861	.0624	.0387	.0825	.0656	.0487	.0319
76	.0150	.1165	.0911	.0657	.0404	.0881	.0699	.0516	.0333
77	.0150	.1237	.0965	.0693	.0422	.0943	.0745	.0546	.0348
78	.0150	.1314	.1023	.0732	.0441	.1009	.0794	.0580	.0365
79	.0150	.1396	.1085	.0773	.0462	.1081	.0848	.0616	.0383
80	.0150	.1485	.1151	.0817	.0484	.1159	.0907	.0655	.0402
81	.0150	.1580	.1222	.0865	.0507	.1244	.0971	.0697	.0424
82	.0150	.1681	.1298	.0915	.0533	.1336	.1039	.0743	.0446
83	.0150	.1790	.1380	.0970	.0560	.1435	.1114	.0793	.0471
84	.0150	.1906	.1467	.1028	.0589	.1543	.1195	.0846	.0498
85	.0150	.2031	.1560	.1090	.0620	.1660	.1282	.0905	.0527
86	.0150	.2164	.1661	.1157	.0654	.1786	.1377	.0968	.0559
87	.0150	.2307	.1768	.1228	.0689	.1923	.1480	.1037	.0593
88	.0150	.2460	.1883	.1305	.0728	.2072	.1591	.1111	.0630
89	.0150	.2624	.2006	.1387	.0769	.2233	.1712	.1192	.0671
90	.0150	.2800	.2137	.1475	.0812	.2408	.1843	.1279	.0714
91	.0150	.2988	.2278	.1569	.0859	.2597	.1985	.1373	.0762
92	.0150	.3189	.2429	.1669	.0910	.2802	.2139	.1476	.0813
93	.0150	.3405	.2591	.1777	.0964	.3024	.2306	.1587	.0869
94	.0150	.3636	.2764	.1893	.1021	.3265	.2487	.1708	.0929
95	.0150	.3883	.2950	.2016	.1083	.3527	.2682	.1838	.0994

ϖ annuity load charged to the consumer.

Table 2.9: Annuity Yields for the Cohorts (Men): 1980 and 2000

	Mature	1980				2000			
Age	Assets	Fair	$\varpi = .25$	$\varpi = .50$	$\varpi = .75$	Fair	$\varpi = .25$	$\varpi = .50$	$\varpi = .75$
65	.0150	.0376	.0319	.0263	.0206	.0308	.0269	.0229	.0190
66	.0150	.0396	.0334	.0273	.0211	.0324	.0280	.0237	.0193
67	.0150	.0418	.0351	.0284	.0217	.0341	.0293	.0245	.0198
68	.0150	.0442	.0369	.0296	.0223	.0359	.0307	.0255	.0202
69	.0150	.0468	.0388	.0309	.0229	.0380	.0322	.0265	.0207
70	.0150	.0496	.0409	.0323	.0236	.0402	.0339	.0276	.0213
71	.0150	.0527	.0432	.0338	.0244	.0426	.0357	.0288	.0219
72	.0150	.0560	.0458	.0355	.0253	.0453	.0377	.0302	.0226
73	.0150	.0597	.0485	.0373	.0262	.0483	.0400	.0316	.0233
74	.0150	.0636	.0515	.0393	.0272	.0515	.0424	.0333	.0241
75	.0150	.0680	.0547	.0415	.0282	.0551	.0450	.0350	.0250
76	.0150	.0727	.0583	.0438	.0294	.0590	.0480	.0370	.0260
77	.0150	.0778	.0621	.0464	.0307	.0632	.0512	.0391	.0271
78	.0150	.0834	.0663	.0492	.0321	.0679	.0547	.0415	.0282
79	.0150	.0895	.0709	.0523	.0336	.0731	.0585	.0440	.0295
80	.0150	.0961	.0759	.0556	.0353	.0787	.0628	.0469	.0309
81	.0150	.1034	.0813	.0592	.0371	.0849	.0674	.0500	.0325
82	.0150	.1112	.0872	.0631	.0391	.0917	.0725	.0533	.0342
83	.0150	.1198	.0936	.0674	.0412	.0992	.0781	.0571	.0360
84	.0150	.1291	.1006	.0721	.0435	.1073	.0843	.0612	.0381
85	.0150	.1393	.1082	.0771	.0461	.1163	.0910	.0657	.0403
86	.0150	.1504	.1165	.0827	.0488	.1262	.0984	.0706	.0428
87	.0150	.1624	.1256	.0887	.0519	.1370	.1065	.0760	.0455
88	.0150	.1755	.1354	.0953	.0551	.1488	.1154	.0819	.0485
89	.0150	.1898	.1461	.1024	.0587	.1619	.1251	.0884	.0517
90	.0150	.2054	.1578	.1102	.0626	.1761	.1359	.0956	.0553
91	.0150	.2223	.1705	.1187	.0668	.1918	.1476	.1034	.0592
92	.0150	.2408	.1843	.1279	.0714	.2090	.1605	.1120	.0635
93	.0150	.2609	.1994	.1379	.0765	.2279	.1747	.1214	.0682
94	.0150	.2828	.2158	.1489	.0819	.2486	.1902	.1318	.0734
95	.0150	.3066	.2337	.1608	.0879	.2713	.2072	.1431	.0791

ϖ annuity load charged to the consumer.

Table 2.10: Annuity Yields for the Cohorts (Women): 1940 and 1960

Age	Mature Assets	1940 Fair	1940 $\varpi = .25$	1940 $\varpi = .50$	1940 $\varpi = .75$	1960 Fair	1960 $\varpi = .25$	1960 $\varpi = .50$	1960 $\varpi = .75$
65	.0150	.0476	.0394	.0313	.0231	.0345	.0296	.0248	.0199
66	.0150	.0500	.0413	.0325	.0238	.0363	.0310	.0257	.0203
67	.0150	.0527	.0433	.0338	.0244	.0383	.0325	.0266	.0208
68	.0150	.0555	.0454	.0353	.0251	.0405	.0341	.0277	.0214
69	.0150	.0586	.0477	.0368	.0259	.0428	.0359	.0289	.0220
70	.0150	.0619	.0502	.0385	.0267	.0454	.0378	.0302	.0226
71	.0150	.0655	.0529	.0402	.0276	.0482	.0399	.0316	.0233
72	.0150	.0693	.0557	.0422	.0286	.0513	.0422	.0332	.0241
73	.0150	.0734	.0588	.0442	.0296	.0547	.0448	.0348	.0249
74	.0150	.0779	.0621	.0464	.0307	.0584	.0475	.0367	.0258
75	.0150	.0826	.0657	.0488	.0319	.0624	.0505	.0387	.0268
76	.0150	.0877	.0696	.0514	.0332	.0668	.0538	.0409	.0279
77	.0150	.0933	.0737	.0541	.0346	.0716	.0575	.0433	.0292
78	.0150	.0992	.0781	.0571	.0360	.0769	.0614	.0459	.0305
79	.0150	.1056	.0829	.0603	.0376	.0826	.0657	.0488	.0319
80	.0150	.1124	.0881	.0637	.0394	.0889	.0704	.0519	.0335
81	.0150	.1198	.0936	.0674	.0412	.0957	.0756	.0554	.0352
82	.0150	.1278	.0996	.0714	.0432	.1032	.0812	.0591	.0371
83	.0150	.1363	.1060	.0756	.0453	.1114	.0873	.0632	.0391
84	.0150	.1455	.1129	.0802	.0476	.1204	.0940	.0677	.0413
85	.0150	.1554	.1203	.0852	.0501	.1302	.1014	.0726	.0438
86	.0150	.1660	.1283	.0905	.0528	.1409	.1094	.0779	.0465
87	.0150	.1775	.1368	.0962	.0556	.1525	.1182	.0838	.0494
88	.0150	.1898	.1461	.1024	.0587	.1653	.1277	.0902	.0526
89	.0150	.2030	.1560	.1090	.0620	.1793	.1382	.0971	.0561
90	.0150	.2173	.1667	.1161	.0656	.1945	.1496	.1048	.0599
91	.0150	.2326	.1782	.1238	.0694	.2112	.1621	.1131	.0640
92	.0150	.2491	.1906	.1320	.0735	.2294	.1758	.1222	.0686
93	.0150	.2668	.2039	.1409	.0780	.2493	.1907	.1322	.0736
94	.0150	.2859	.2182	.1505	.0827	.2711	.2070	.1430	.0790
95	.0150	.3064	.2336	.1607	.0879	.2948	.2249	.1549	.0850

ϖ annuity load charged to the consumer.

Table 2.11: Annuity Yields for the Cohorts (Women): 1980 and 2000

	Mature	1980				2000			
Age	Assets	Fair	$\varpi = .25$	$\varpi = .50$	$\varpi = .75$	Fair	$\varpi = .25$	$\varpi = .50$	$\varpi = .75$
65	.0150	.0262	.0234	.0206	.0178	.0213	.0197	.0181	.0166
66	.0150	.0274	.0243	.0212	.0181	.0221	.0203	.0185	.0168
67	.0150	.0288	.0253	.0219	.0184	.0229	.0210	.0190	.0170
68	.0150	.0302	.0264	.0226	.0188	.0239	.0217	.0195	.0172
69	.0150	.0319	.0277	.0234	.0192	.0250	.0225	.0200	.0175
70	.0150	.0337	.0290	.0244	.0197	.0262	.0234	.0206	.0178
71	.0150	.0357	.0306	.0254	.0202	.0276	.0245	.0213	.0182
72	.0150	.0380	.0322	.0265	.0207	.0292	.0256	.0221	.0185
73	.0150	.0405	.0341	.0277	.0214	.0309	.0269	.0229	.0190
74	.0150	.0432	.0362	.0291	.0221	.0328	.0284	.0239	.0195
75	.0150	.0463	.0385	.0307	.0228	.0350	.0300	.0250	.0200
76	.0150	.0497	.0410	.0323	.0237	.0375	.0319	.0262	.0206
77	.0150	.0534	.0438	.0342	.0246	.0402	.0339	.0276	.0213
78	.0150	.0576	.0470	.0363	.0257	.0433	.0362	.0292	.0221
79	.0150	.0622	.0504	.0386	.0268	.0468	.0389	.0309	.0230
80	.0150	.0673	.0542	.0412	.0281	.0507	.0418	.0329	.0239
81	.0150	.0730	.0585	.0440	.0295	.0551	.0451	.0350	.0250
82	.0150	.0793	.0632	.0471	.0311	.0600	.0487	.0375	.0262
83	.0150	.0862	.0684	.0506	.0328	.0655	.0529	.0403	.0276
84	.0150	.0939	.0742	.0545	.0347	.0717	.0575	.0434	.0292
85	.0150	.1025	.0806	.0587	.0369	.0787	.0627	.0468	.0309
86	.0150	.1120	.0877	.0635	.0392	.0865	.0686	.0507	.0329
87	.0150	.1225	.0956	.0687	.0419	.0952	.0752	.0551	.0351
88	.0150	.1341	.1043	.0745	.0448	.1051	.0826	.0600	.0375
89	.0150	.1470	.1140	.0810	.0480	.1161	.0908	.0656	.0403
90	.0150	.1613	.1247	.0881	.0516	.1285	.1001	.0718	.0434
91	.0150	.1771	.1366	.0960	.0555	.1424	.1106	.0787	.0469
92	.0150	.1946	.1497	.1048	.0599	.1581	.1223	.0865	.0508
93	.0150	.2141	.1643	.1145	.0648	.1756	.1355	.0953	.0552
94	.0150	.2356	.1805	.1253	.0702	.1953	.1502	.1052	.0601
95	.0150	.2595	.1984	.1373	.0761	.2174	.1668	.1162	.0656

ϖ annuity load charged to the consumer.

Table 2.12: Pension Benefits (Men): Unfunded Social Security

γ	r^*	Cohort			
		1940	**1960**	**1980**	**2000**
.75	.0150	34.656	21.709	17.510	14.326
	.0213	33.379	20.909	16.865	13.798
2	.0150	34.656	21.709	17.510	14.326
	.0400	31.041	19.444	15.683	12.831
5	.0150	34.656	21.709	17.510	14.326
	.0850	28.234	17.686	14.265	11.671

Parameters: payroll tax τ equals .08; population growth rate $n = .015$; $\delta = .01$; $J = 65$ and a CES Production Function with $\{A = 2093, \theta = \frac{1}{3}, \alpha = -0.131855\}$.

We have used in this table the following formulae:

- *Golden Rule*: $r^* = n$.
- *Modified Golden Rule*: $r^* = \delta + \gamma n$.
- *Pension Benefit*: $b = \tau w^* \frac{\int_0^J \Omega(s,x) \mathrm{e}^{-ns} ds}{\int_J^T \Omega(s,x) \mathrm{e}^{-ns} ds}$, where $x := \{1940, 1960, 1980, 2000\}$.
- *Wage*: $w^* = \phi(r^*)$.

Tables 2.12 and 2.17 below report pension benefits that correspond to an individual (either men or women) who belongs to an economy with a specific survival probability. Let note, as both factor prices are under stationary conditions, that there exists a negative relationship between the pension benefit and the cohort that our individual belongs. That is, the greater the life expectancy is, the lower the benefit becomes. According to the results reported in tables 2.12 and 2.17, we find that the higher the stationary interest rate is, the lower both the stationary wage and the pension benefit is. Consequently, attending to proposition 2.1 we expect that, under stationary conditions, the greater the interest rate is, the greater the willingness of purchasing annuities becomes.

Table 2.13: INITIAL WEALTH PATH FROM 65 TO 85 YEARS OLD. COHORT (MEN) 1940

$\gamma = .75$, $r^* = .0150$, $b = 34.656$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	0	0	0	0	0	0	0
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = .75$ and $r^* = .02125$, $b = 33.379$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	0	0	0	0	0	0	0
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0150$, $b = 34.656$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	0	0	0	0	0	0	0
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0400$, $b = 31.041$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	278.516	0	146.632	0	112.941	0	112.941
$k(75)$	0	252.956	0	0	0	0	0	0
$k(80)$	0	224.988	0	0	0	0	0	0
$k(85)$	0	196.299	0	0	0	0	0	0

$\gamma = 5$ and $r^* = .0150$, $b = 34.656$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	174344	0	108.672	0	108.672	0	108.672
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 5$ and $r^* = .0850$, $b = 28.234$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	148.051	286.396	145.641	283.755	142.682	281.002	138.094	273.315
$k(75)$	142.047	267.052	136.168	260.042	128.995	252.587	117.944	242.601
$k(80)$	132.683	243.361	122.228	230.148	109.632	216.018	62.089	197.106
$k(85)$	120.920	217.059	104.869	195.924	85.985	173.571	0	143.836

ϖ annuity load charged to the consumer.

Figure 2.3: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0150,\, b = 34.656\}$

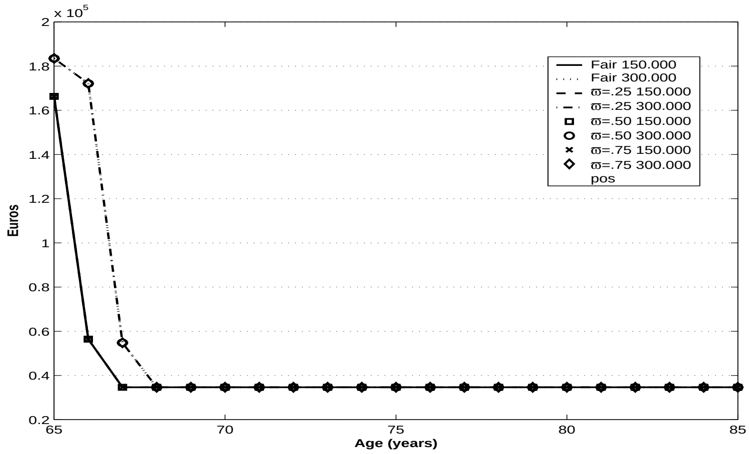


Figure 2.4: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0150,\, b = 34.656\}$

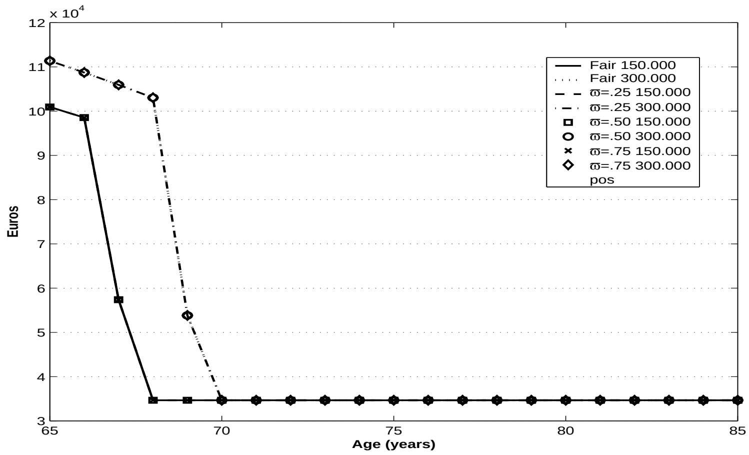


Figure 2.5: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0150,\, b = 34.656\}$

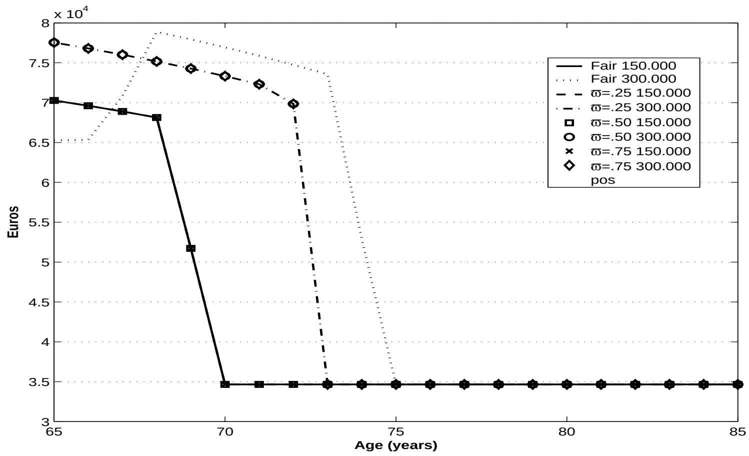


Figure 2.6: Consumption Trajectories. Case: $\left\{\gamma = .75,\, r^* = .0213,\, b = 33.379\right\}$

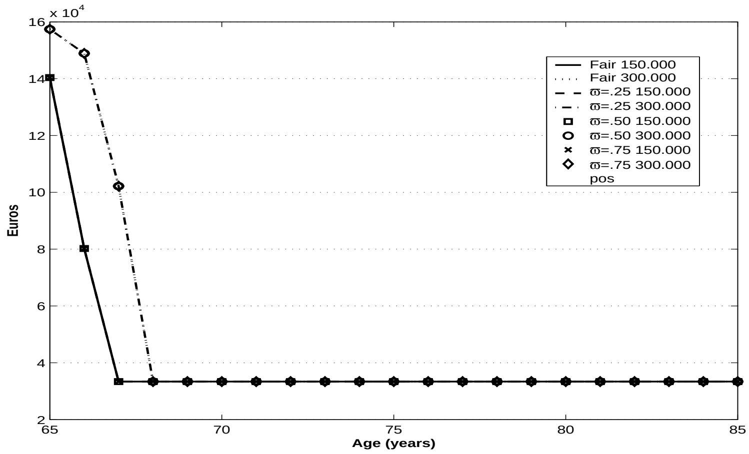


Figure 2.7: Consumption Trajectories. Case: $\left\{\gamma = 2,\, r^* = .0400,\, b = 31.041\right\}$

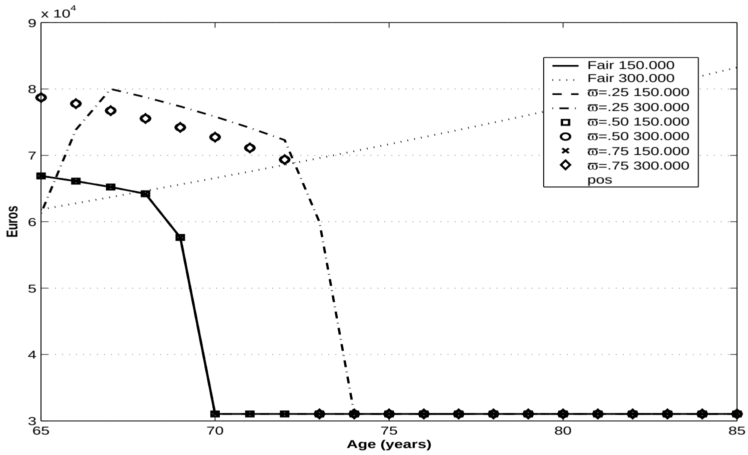


Figure 2.8: Consumption Trajectories. Case: $\left\{\gamma = 5,\, r^* = .0850,\, b = 28.234\right\}$

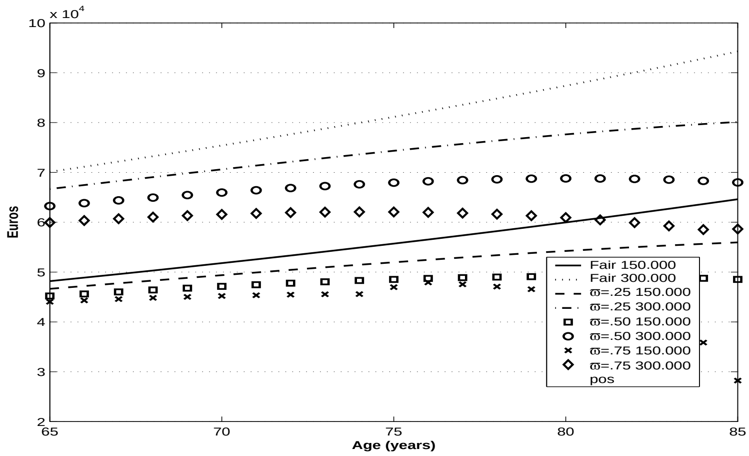


Table 2.14: Initial Wealth Path from 65 to 85 years old. Cohort (Men) 1960

$\gamma = .75$ and $r^* = .0150$, $b = 21.709$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	0	0	0	0	0	0	0
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = .75$ and $r^* = .02125$, $b = 20.909$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	0	0	0	0	0	0	0
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0150$, $b = 21.709$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	93.688	0	93.688	0	93.688	0	93.688
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0400$, $b = 19.444$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	141.449	274.009	64.663	267.730	64.663	209.149	64.663	189.765
$k(75)$	59.532	243.741	0	223.548	0	82.992	0	67.916
$k(80)$	0	211.147	0	177.619	0	0	0	0
$k(85)$	0	178.293	0	0	0	0	0	0

$\gamma = 5$ and $r^* = .0150$, $b = 21.709$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	52.074	244.794	52.074	242.432	52.074	179.147	52.074	179.147
$k(75)$	0	195.202	0	130.335	0	59.617	0	59.617
$k(80)$	0	152.098	0	1.599	0	0	0	0
$k(85)$	0	115.915	0	0	0	0	0	0

$\gamma = 5$ and $r^* = .0850$, $b = 17.686$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	145.494	284.183	145.043	284.303	144.549	284.789	143.618	285.495
$k(75)$	136.802	261.701	135.076	260.527	133.124	259.944	129.942	259.455
$k(80)$	124.651	234.085	120.637	229.856	116.058	226.218	108.997	221.994
$k(85)$	110.148	203.455	102.786	194.298	94.423	185.559	77.366	174.934

ϖ annuity load charged to the consumer.

Figure 2.9: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0150,\, b = 21.709\}$

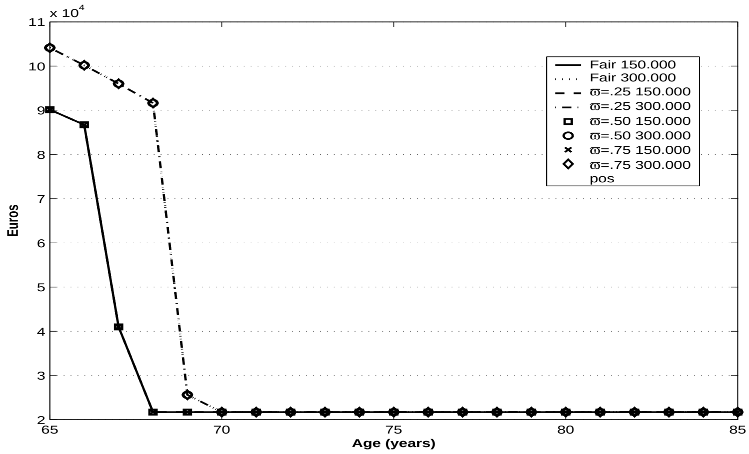


Figure 2.10: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0150,\, b = 21.709\}$

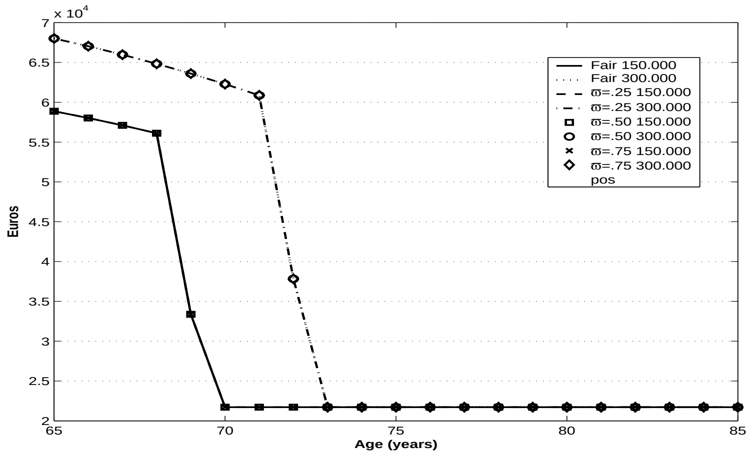


Figure 2.11: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0150,\, b = 21.709\}$

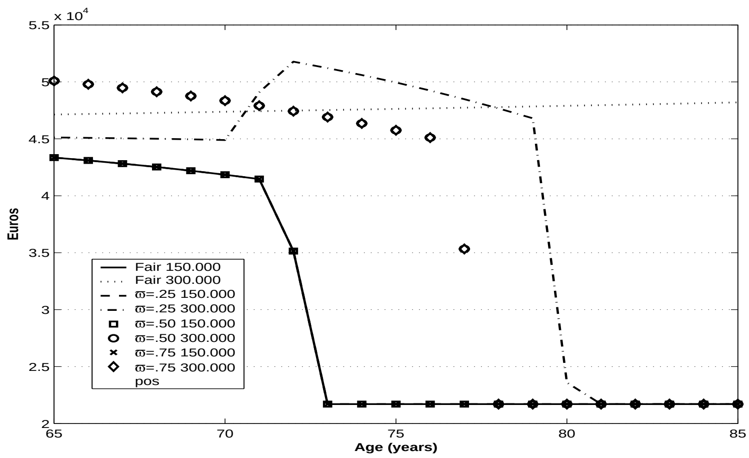


Figure 2.12: Consumption Trajectories. Case: $\{\gamma = .75,\ r^* = .0213,\ b = 20.909\}$

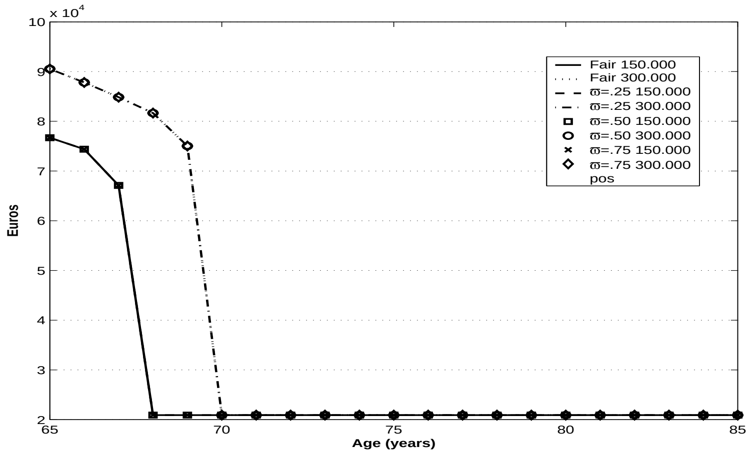


Figure 2.13: Consumption Trajectories. Case: $\{\gamma = 2,\ r^* = .0400,\ b = 19.444\}$

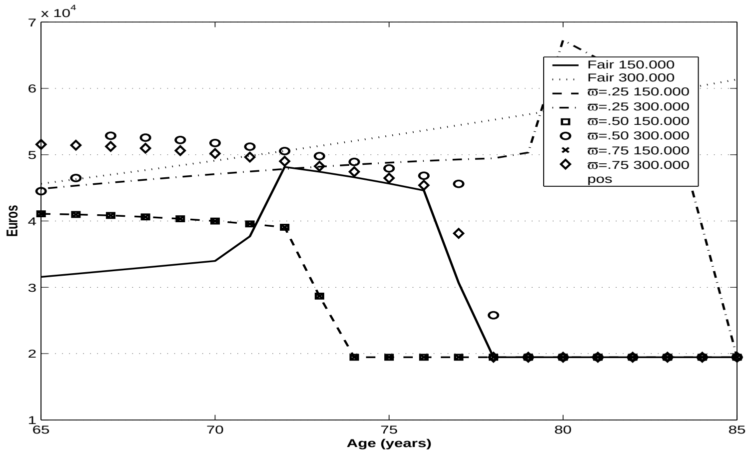


Figure 2.14: Consumption Trajectories. Case: $\{\gamma = 5,\ r^* = .0850,\ b = 17.686\}$

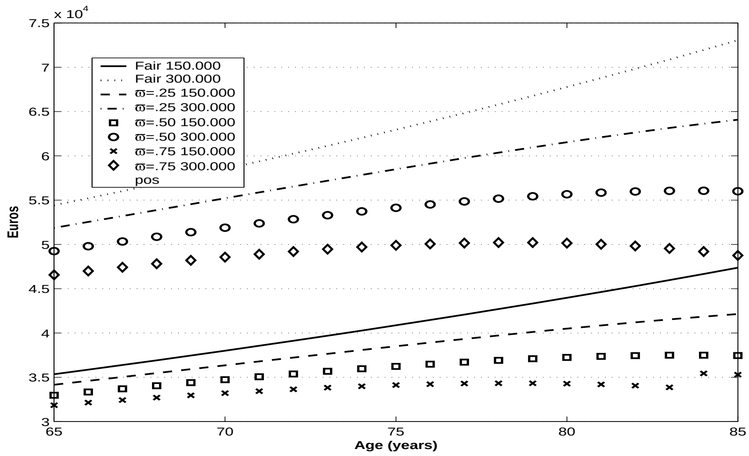


Table 2.15: Initial Wealth Path from 65 to 85 years old. Cohort (Men) 1980

$\gamma = .75$ and $r^* = .0150$, $b = 17.510$	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	28.431	0	28.431	0	28.431	0	28.431
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0
$\gamma = .75$ and $r^* = .02125$, $b = 16.865$	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	73.855	0	73.855	0	73.855	0	73.855
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0
$\gamma = 2$ and $r^* = .0150$, $b = 17.510$	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	22.867	141.825	22.867	141.825	22.867	141.825	22.867	141.825
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0
$\gamma = 2$ and $r^* = .0400$, $b = 15.683$	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	141.267	274.723	88.944	268.382	88.944	260.692	88.944	217.796
$k(75)$	128.964	244.647	17.751	230.334	17.751	197.241	17.751	121.787
$k(80)$	114.094	211.684	0	187.999	0	75.451	0	19.485
$k(85)$	96.053	177.998	0	144.121	0	0	0	0
$\gamma = 5$ and $r^* = .0150$, $b = 17.510$	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	75.962	246.138	75.962	245.167	75.962	204.777	75.962	204.777
$k(75)$	1.680	196.964	1.680	194.066	1.680	108.688	1.680	108.688
$k(80)$	0	153.598	0	147.992	0	15.226	0	15.226
$k(85)$	0	116.756	0	41.279	0	0	0	0
$\gamma = 5$ and $r^* = .0850$, $b = 14.265$	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	145.865	285.805	146.053	286.850	146.328	288.365	146.495	290.379
$k(75)$	137.411	264.388	137.128	265.341	136.934	267.165	136.300	269.819
$k(80)$	125.228	237.060	123.570	236.323	121.861	236.670	119.025	237.767
$k(85)$	110.375	205.898	106.296	201.605	101.874	198.357	95.211	195.165

ϖ annuity load charged to the consumer.

Figure 2.15: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0150,\, b = 17.510\}$

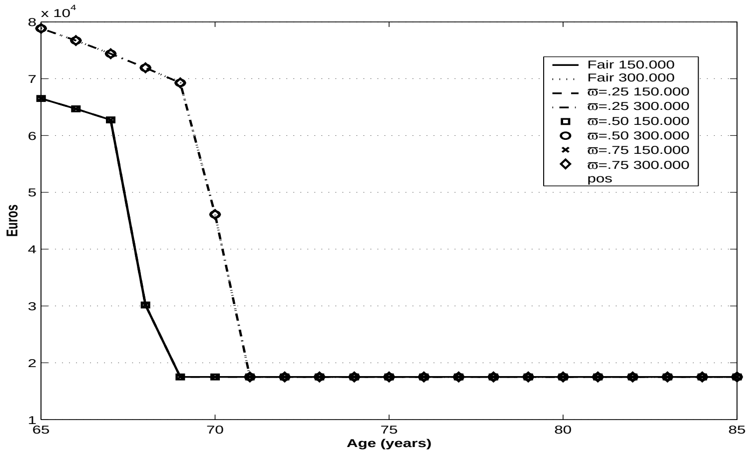


Figure 2.16: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0150,\, b = 17.510\}$

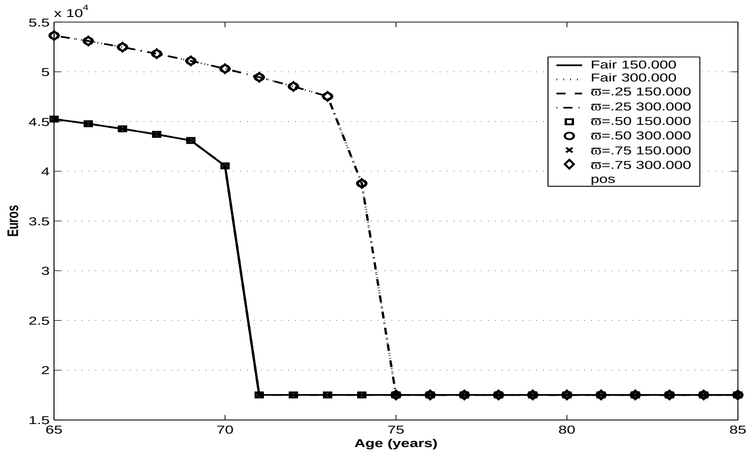


Figure 2.17: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0150,\, b = 17.510\}$

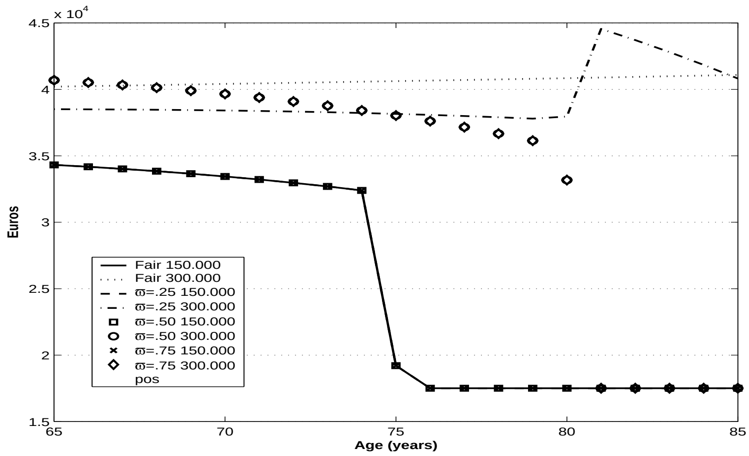


Figure 2.18: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0213,\, b = 16.865\}$

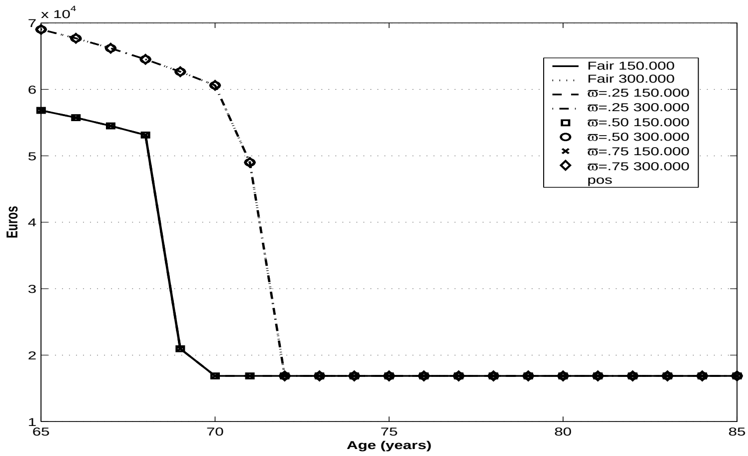


Figure 2.19: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0400,\, b = 15.683\}$

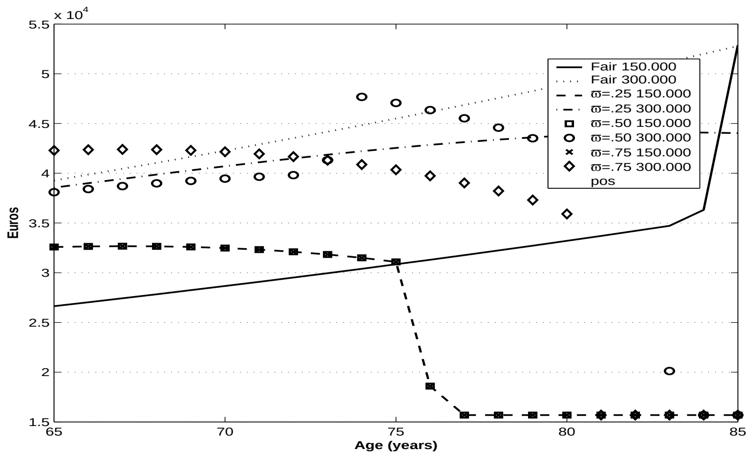


Figure 2.20: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0850,\, b = 14.265\}$

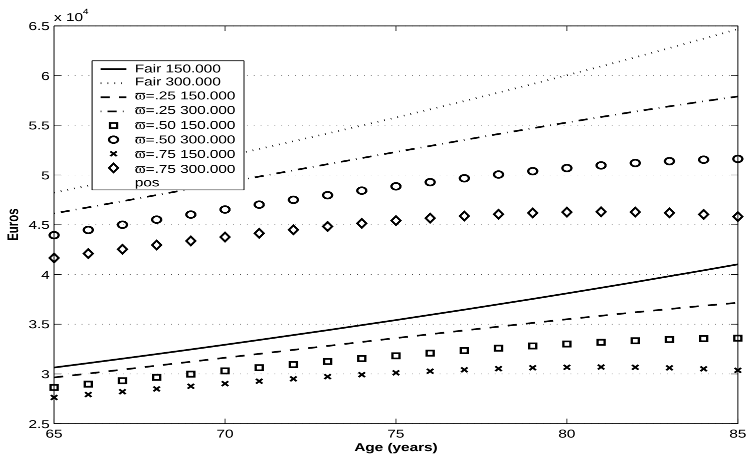


Table 2.16: Initial Wealth Path from 65 to 85 years old. Cohort (Men) 2000

$\gamma = .75$ and $r^* = .0150$, $b = 14.326$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	94.633	0	94.633	0	94.633	0	94.633
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = .75$ and $r^* = .02125$, $b = 13.798$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	19.086	130.630	19.086	130.630	19.086	130.630	19.086	130.630
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0150$, $b = 14.326$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	54.211	249.358	54.211	176.191	54.211	176.191	54.211	176.191
$k(75)$	0	120.609	0	54.532	0	54.532	0	54.532
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0400$, $b = 12.831$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	141.185	275.442	137.974	271.504	106.127	266.682	106.127	238.368
$k(75)$	128.581	245.347	104.211	235.946	51.732	224.722	51.732	162.013
$k(80)$	113.144	211.610	30.767	195.222	0	169.805	0	75.401
$k(85)$	96.140	176.565	0	151.994	0	52.682	0	0

$\gamma = 5$ and $r^* = .0150$, $b = 14.326$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	92.296	247.287	92.296	247.435	92.296	247.259	92.296	222.576
$k(75)$	33.230	198.263	33.230	197.560	33.230	177.343	33.230	142.940
$k(80)$	0	154.322	0	151.839	0	91.262	0	63.454
$k(85)$	0	116.526	0	111.553	0	9.075	0	0

$\gamma = 5$ and $r^* = .0850$, $b = 11.671$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	146.359	287.503	147.032	289.269	147.867	291.553	148.791	294.483
$k(75)$	138.131	267.032	138.990	269.727	140.139	273.462	141.356	278.473
$k(80)$	125.756	239.634	126.016	241.857	126.589	245.512	126.960	250.822
$k(85)$	110.257	207.373	108.901	207.283	107.712	208.750	105.661	211.705

ϖ annuity load charged to the consumer.

Figure 2.21: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0150,\, b = 14.326\}$

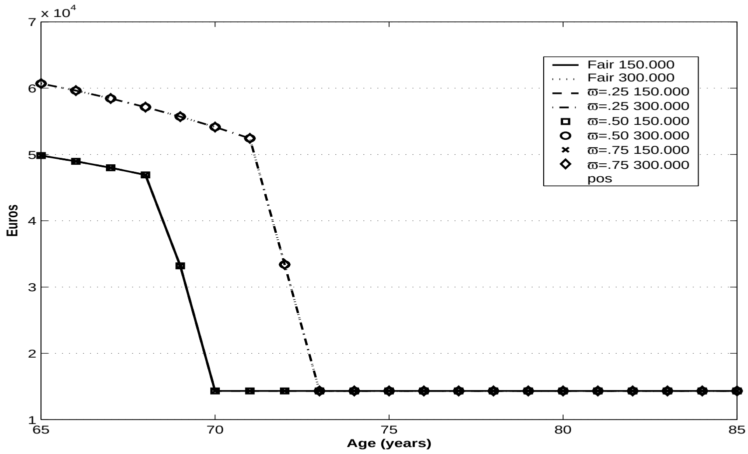


Figure 2.22: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0150,\, b = 14.326\}$

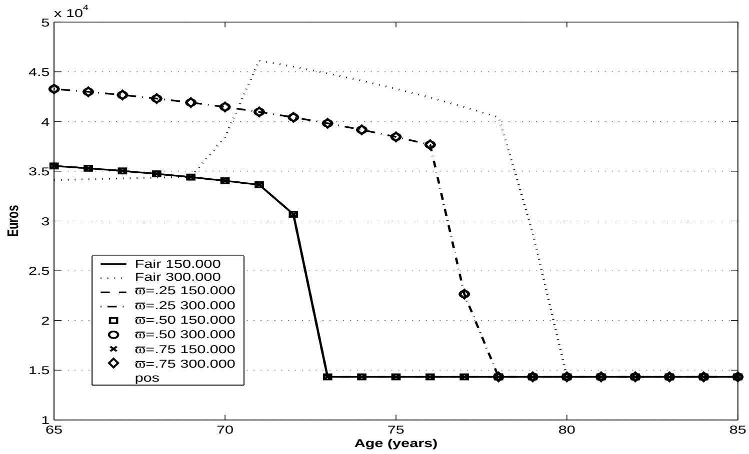


Figure 2.23: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0150,\, b = 14.326\}$

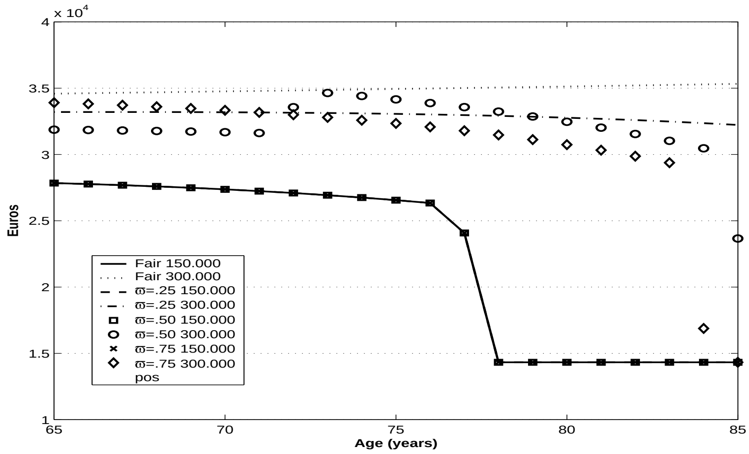


Figure 2.24: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0213,\, b = 13.798\}$

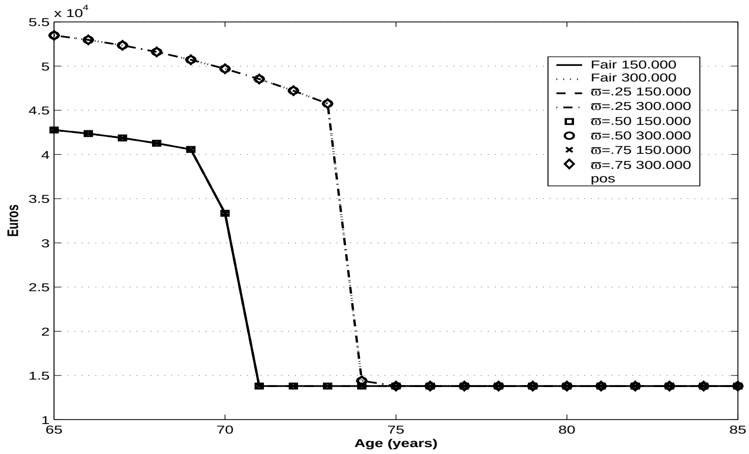


Figure 2.25: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0400,\, b = 12.831\}$

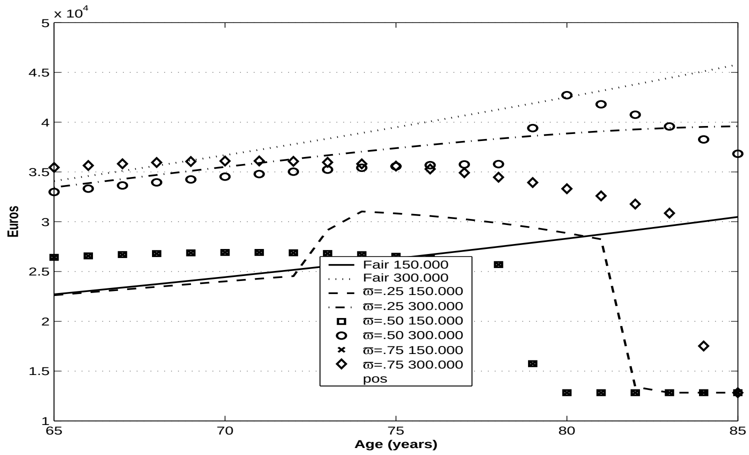


Figure 2.26: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0850,\, b = 11.671\}$

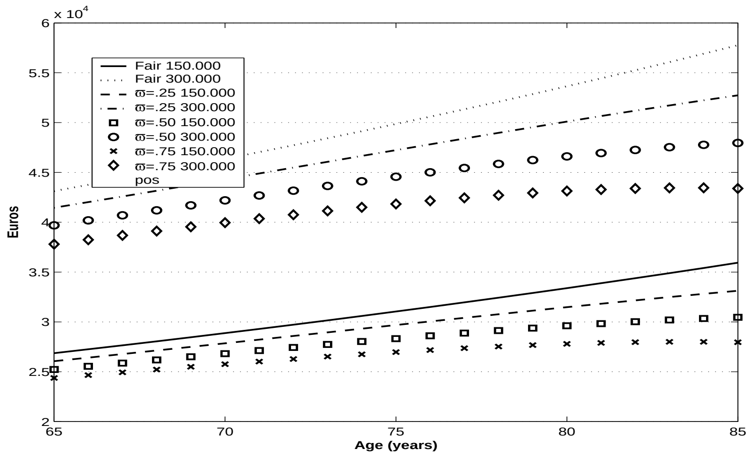


Table 2.17: UNFUNDED PENSION BENEFITS (WOMEN)

γ	r	Cohort 1940	1960	1980	2000
.75	.0150	22.980	16.031	12.463	10.418
	.0213	22.134	15.441	12.004	10.034
2	.0150	22.980	16.031	12.463	10.418
	.0400	20.583	14.359	11.163	9.331
5	.0150	22.980	16.031	12.463	10.418
	.0850	18.722	13.061	10.153	8.487

Parameters: payroll tax τ equals .08; population growth rate $n = .015$; $\delta = .01$; $J = 65$ and a CES Production Function with $\{A = 2093, \theta = \frac{1}{3}, \alpha = -0.131855\}$.

We have used in this table the following formulae:

- *Golden Rule*: $r^* = n$.
- *Modified Golden Rule*: $r^* = \delta + \gamma n$.
- *Pension Benefit*: $b = \tau w^* \frac{\int_0^J \Omega(s,x) e^{-ns} ds}{\int_J^T \Omega(s,x) e^{-ns} ds}$, where $x := \{1940, 1960, 1980, 2000\}$.
- *Wage*: $w^* = \phi(r^*)$.

Table 2.18: Initial Wealth Path from 65 to 85 years old. Cohort (Women) 1940

$\gamma = .75$ and $r^* = .0150$, $b = 22.980$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	0	0	0	0	0	0	0
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = .75$ and $r^* = .02125$, $b = 22.134$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	0	0	0	0	0	0	0
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0150$, $b = 22.980$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	90.830	0	90.830	0	90.830	0	90.830
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0400$, $b = 20.583$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	143.904	278.172	63.762	267.894	63.762	225.193	63.762	189.509
$k(75)$	134.330	252.095	0	229.971	0	96.826	0	68.755
$k(80)$	36.989	223.338	0	188.270	0	0	0	0
$k(85)$	0	193.593	0	39.633	0	0	0	0

$\gamma = 5$ and $r^* = .0150$, $b = 22.980$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	53.300	248.151	53.300	245.152	53.300	181.204	53.300	181.204
$k(75)$	0	201.351	0	192.524	0	64.484	0	64.484
$k(80)$	0	160.274	0	55.918	0	0	0	0
$k(85)$	0	125.259	0	0	0	0	0	0

$\gamma = 5$ and $r^* = .0850$, $b = 18.722$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	147.473	287.623	146.750	287.285	145.942	287.256	144.632	287.352
$k(75)$	141.052	269.115	138.814	267.104	136.274	265.577	132.374	263.957
$k(80)$	131.264	245.617	126.559	240.304	121.189	235.460	113.174	229.797
$k(85)$	118.951	218.760	110.773	208.404	101.492	198.368	87.883	186.238

ϖ annuity load charged to the consumer.

Figure 2.27: Consumption Trajectories. Case: $\{\gamma = .75,\ r^* = .0150,\ b = 22.980\}$

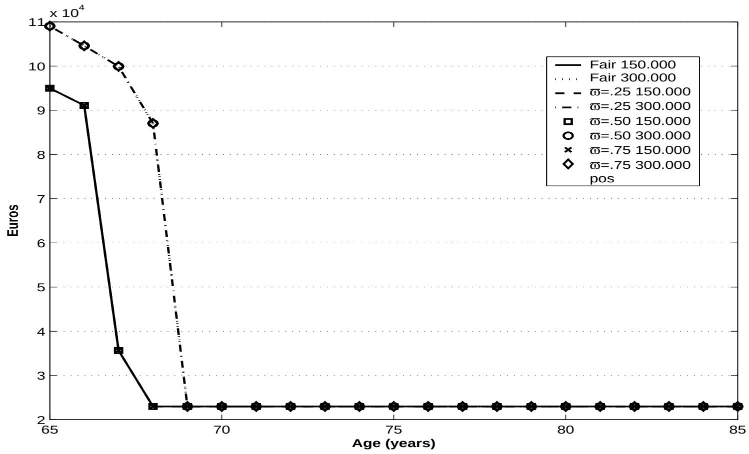


Figure 2.28: Consumption Trajectories. Case: $\{\gamma = 2,\ r^* = .0150,\ b = 22.980\}$

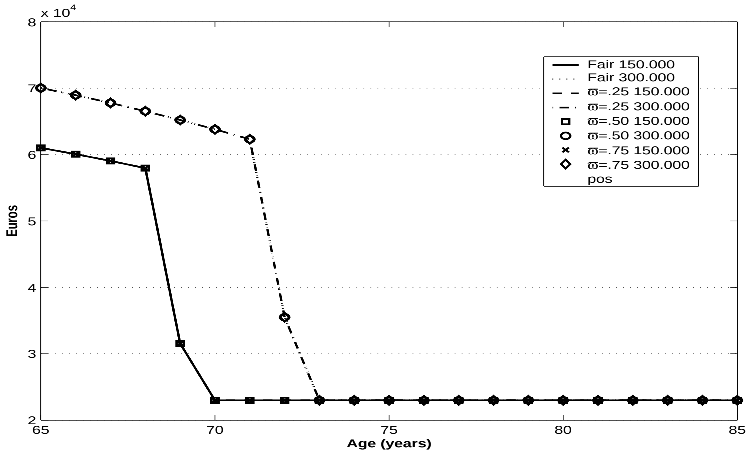


Figure 2.29: Consumption Trajectories. Case: $\{\gamma = 5,\ r^* = .0150,\ b = 22.980\}$

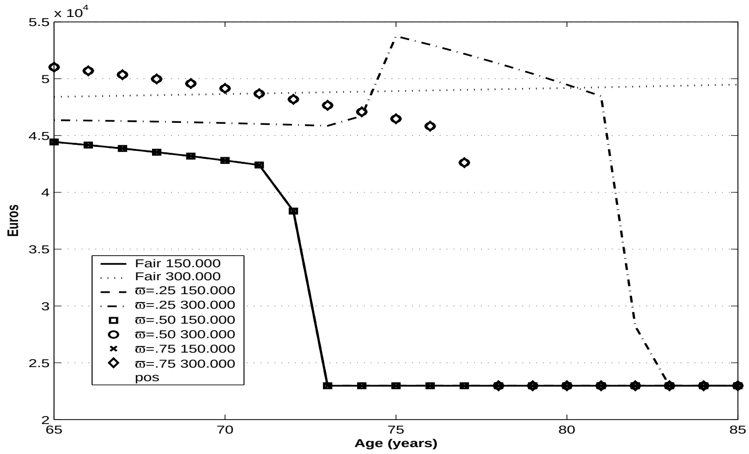


Figure 2.30: Consumption Trajectories. Case: $\{\gamma = .75,\ r^* = .0213,\ b = 22.134\}$

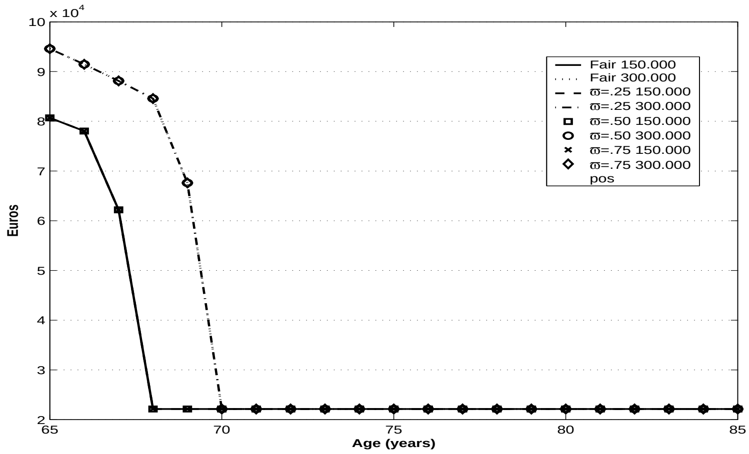


Figure 2.31: Consumption Trajectories. Case: $\{\gamma = 2,\ r^* = .0400,\ b = 20.583\}$

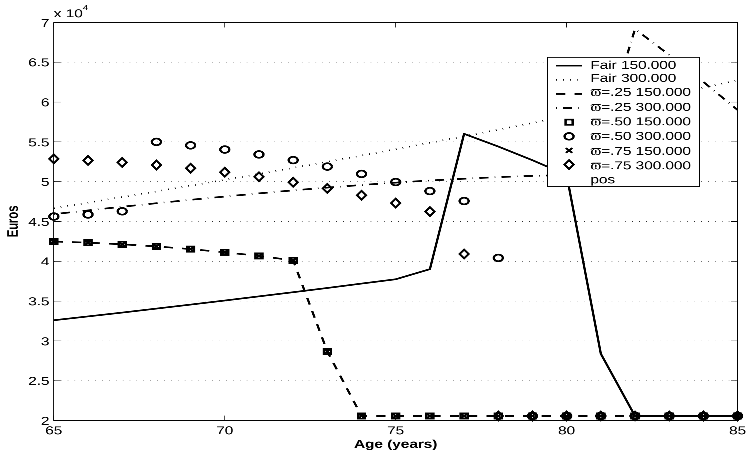


Figure 2.32: Consumption Trajectories. Case: $\{\gamma = 5,\ r^* = .0850,\ b = 18.722\}$

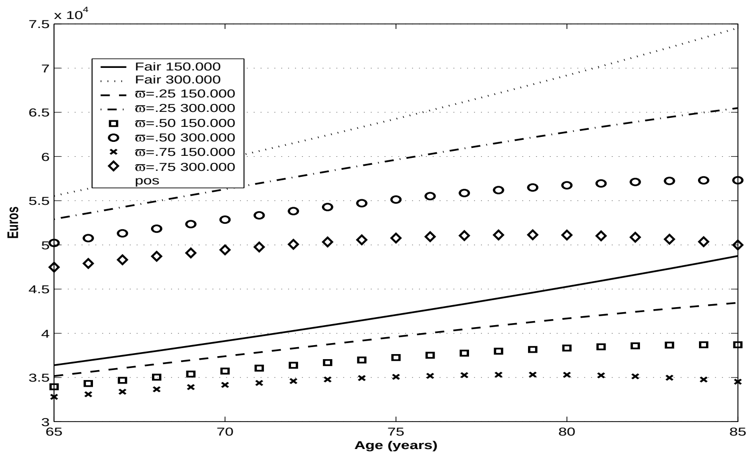


Table 2.19: Initial Wealth Path from 65 to 85 years old. Cohort (Women) 1960

$\gamma = .75$ and $r^* = .0150$, $b = 16.031$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	58.743	0	58.743	0	58.743	0	58.743
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = .75$ and $r^* = .02125$, $b = 15.441$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	0	99.751	0	99.751	0	99.751	0	99.751
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0150$, $b = 16.031$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	37.168	157.360	37.168	157.360	37.168	157.360	37.168	157.360
$k(75)$	0	20.043	0	20.043	0	20.043	0	20.043
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0400$, $b = 14.359$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	141.038	274.693	117.470	269.467	96.698	263.111	96.698	226.939
$k(75)$	128.420	244.286	49.925	232.230	32.934	217.930	32.934	139.470
$k(80)$	113.149	210.722	0	190.363	0	114.803	0	43.785
$k(85)$	96.480	176.270	0	146.613	0	0	0	0

$\gamma = 5$ and $r^* = .0150$, $b = 16.031$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	83.262	246.355	83.262	245.919	83.262	221.545	83.262	212.642
$k(75)$	15.704	197.031	15.704	195.162	15.704	131.169	15.704	123.735
$k(80)$	0	153.281	0	149.118	0	42.219	0	36.273
$k(85)$	0	115.974	0	85.547	0	0	0	0

$\gamma = 5$ and $r^* = .0850$, $b = 13.061$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	145.908	286.231	146.332	287.630	146.881	289.527	147.420	292.003
$k(75)$	137.365	264.885	137.622	266.662	138.065	269.398	138.313	273.189
$k(80)$	124.922	237.207	124.148	237.829	123.493	239.702	122.145	242.756
$k(85)$	109.646	205.320	106.798	202.911	103.838	201.776	99.265	201.351

ϖ annuity load charged to the consumer.

Figure 2.33: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0150,\, b = 16.031\}$

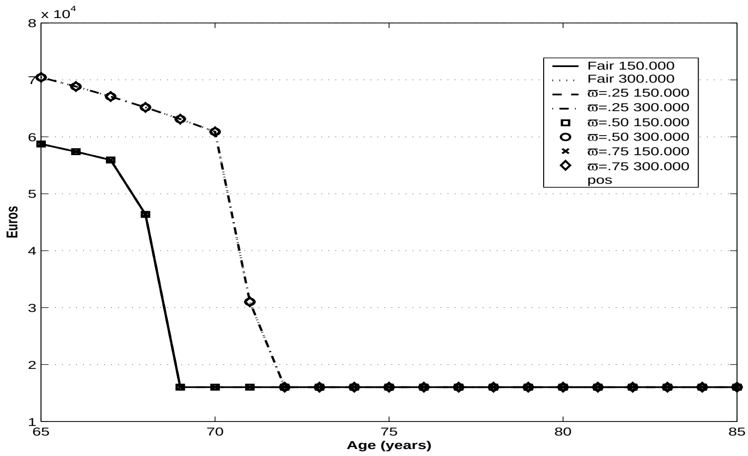


Figure 2.34: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0150,\, b = 16.031\}$

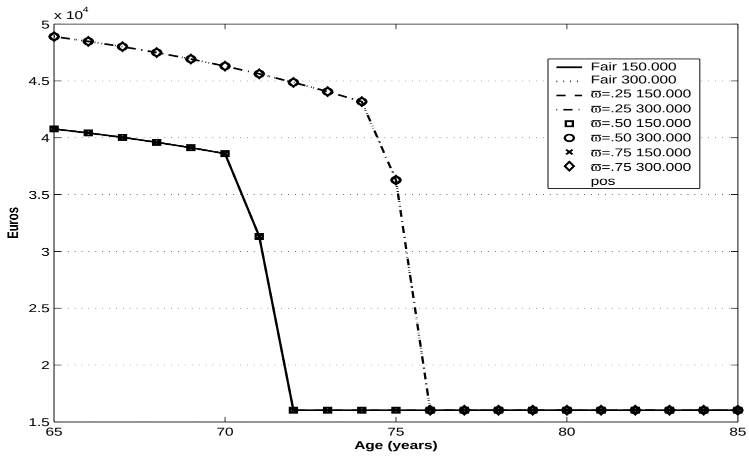


Figure 2.35: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0150,\, b = 16.031\}$

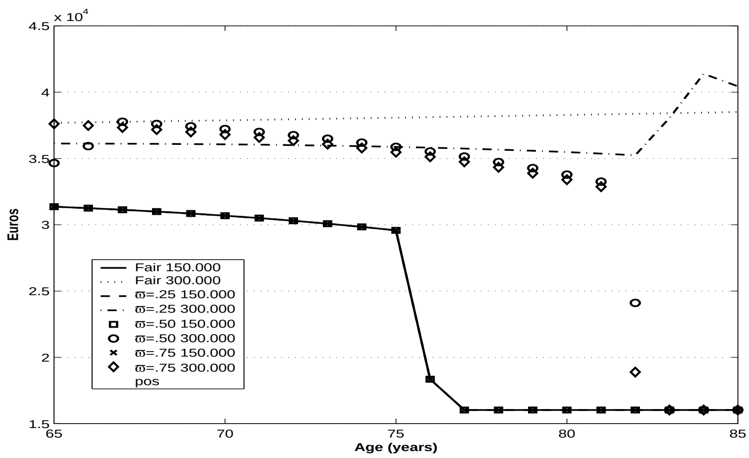


Figure 2.36: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0213,\, b = 15.441\}$

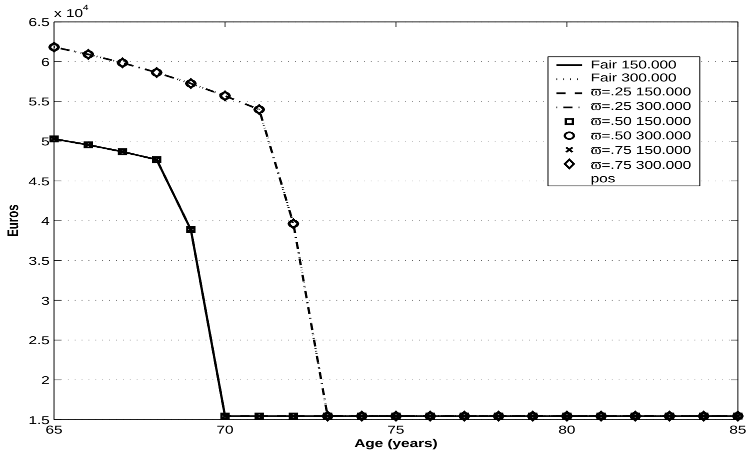


Figure 2.37: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0400,\, b = 14.359\}$

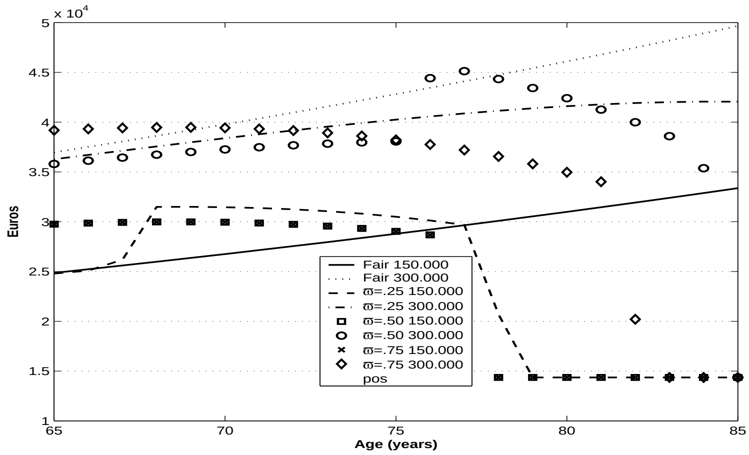


Figure 2.38: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0850,\, b = 13.061\}$

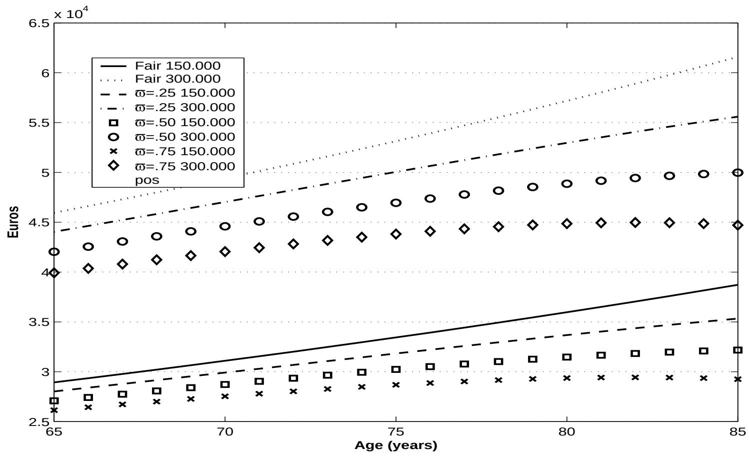


Table 2.20: Initial Wealth Path from 65 to 85 years old. Cohort (Women) 1980

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$\gamma = .75$ and $r^* = .0150$, $b = 12.463$								
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	20.289	131.903	20.289	131.903	20.289	131.903	20.289	131.903
$k(75)$	0	0	0	0	0	0	0	0
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0
$\gamma = .75$ and $r^* = .02125$, $b = 12.004$								
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	46.843	162.749	46.843	162.749	46.843	162.749	46.843	162.749
$k(75)$	0	23.652	0	23.652	0	23.652	0	23.652
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0
$\gamma = 2$ and $r^* = .0150$, $b = 12.463$								
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	70.678	249.541	70.678	194.515	70.678	194.515	70.678	194.515
$k(75)$	0	192.783	0	88.038	0	88.038	0	88.038
$k(80)$	0	62.417	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0
$\gamma = 2$ and $r^* = .0400$, $b = 11.163$								
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	140.907	275.469	138.778	273.082	115.205	270.069	115.205	249.569
$k(75)$	127.695	244.589	122.607	238.419	69.806	230.894	69.806	183.994
$k(80)$	111.323	209.306	71.161	197.793	15.860	184.247	15.860	105.976
$k(85)$	93.180	172.242	0	153.965	0	98.129	0	22.400
$\gamma = 5$ and $r^* = .0150$, $b = 12.463$								
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	100.250	247.569	100.250	248.432	100.250	249.169	100.250	231.261
$k(75)$	48.547	198.013	48.547	198.742	48.547	199.094	48.547	159.560
$k(80)$	0	153.010	0	152.588	0	121.488	0	86.657
$k(85)$	0	113.994	0	111.544	0	42.356	0	15.563
$\gamma = 5$ and $r^* = .0850$, $b = 10.153$								
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	146.590	288.418	147.552	290.630	148.710	293.369	150.044	296.809
$k(75)$	138.203	267.974	139.776	271.799	141.740	276.740	144.016	283.162
$k(80)$	125.159	239.506	126.647	243.694	128.647	249.509	130.922	257.451
$k(85)$	108.517	205.182	108.911	207.888	109.770	212.470	110.519	219.345

ϖ annuity load charged to the consumer.

Figure 2.39: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0150,\, b = 12.463\}$

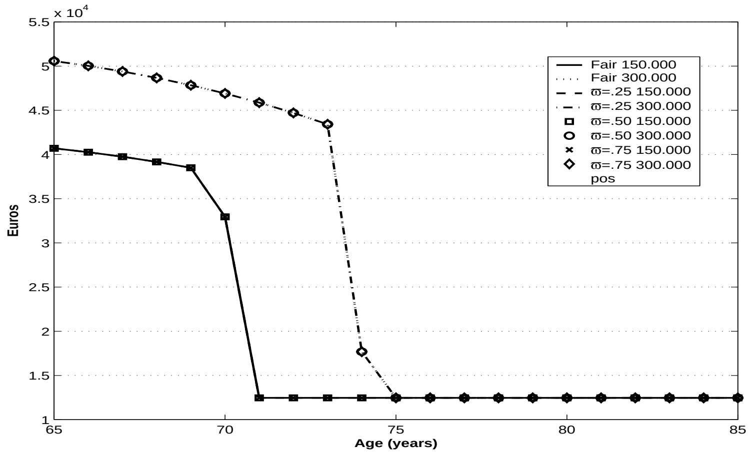


Figure 2.40: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0150,\, b = 12.463\}$

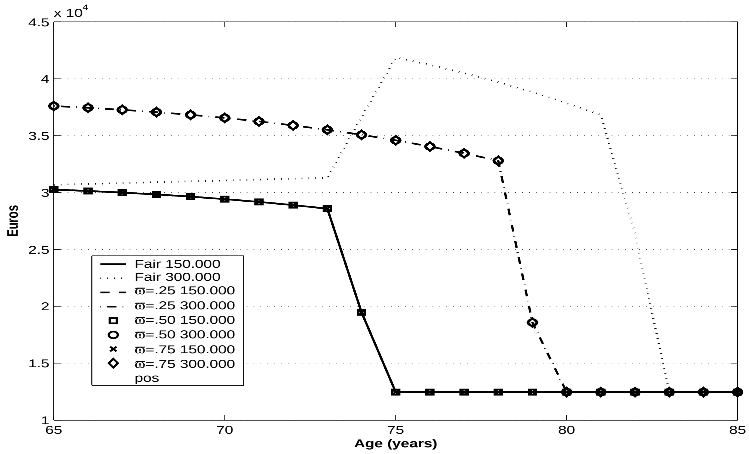


Figure 2.41: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0150,\, b = 12.463\}$

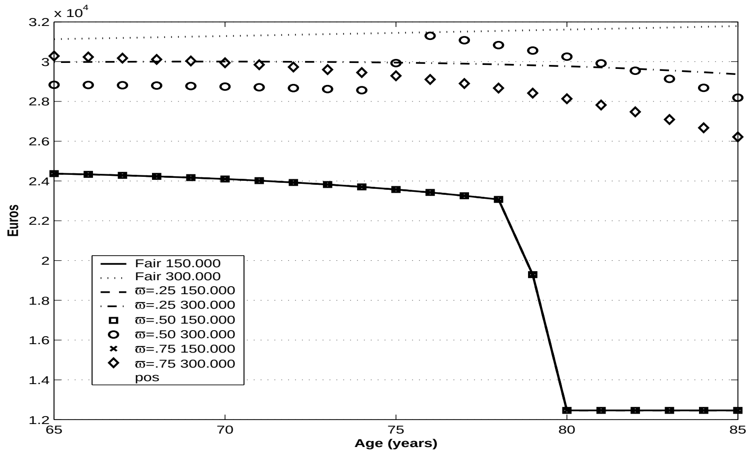


Figure 2.42: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0213,\, b = 12.004\}$

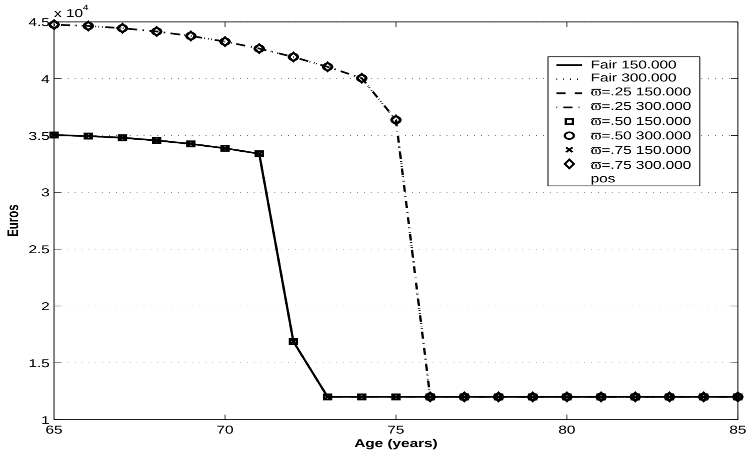


Figure 2.43: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0400,\, b = 11.163\}$

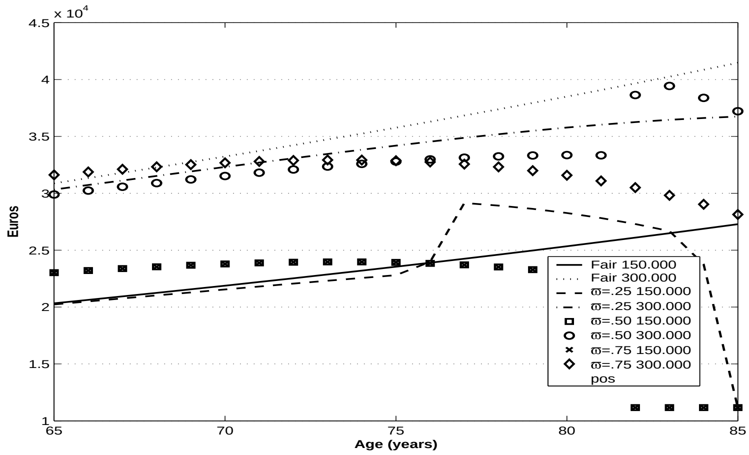


Figure 2.44: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0850,\, b = 10.153\}$

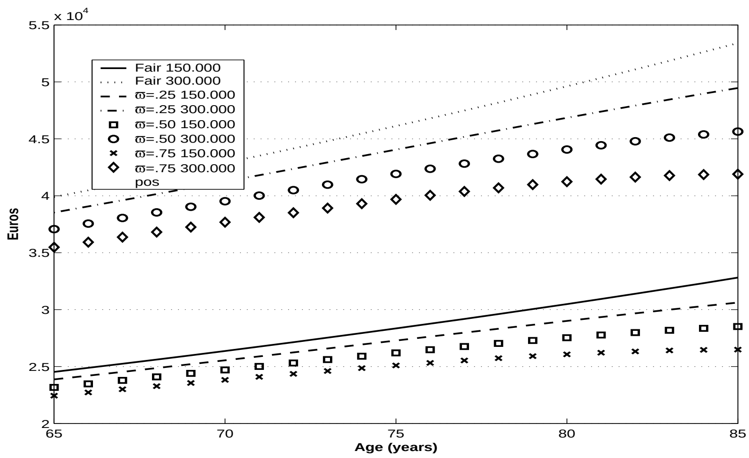


Table 2.21: Initial Wealth Path from 65 to 85 years old. Cohort (Women) 2000

$\gamma = .75$ and $r^* = .0150$, $b = 10.418$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	57.423	175.173	57.423	175.173	57.423	175.173	57.423	175.173
$k(75)$	0	50.200	0	50.200	0	50.200	0	50.200
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = .75$ and $r^* = .02125$, $b = 10.034$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	78.324	200.566	78.324	200.566	78.324	200.566	78.324	200.566
$k(75)$	0	91.363	0	91.363	0	91.363	0	91.363
$k(80)$	0	0	0	0	0	0	0	0
$k(85)$	0	0	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0150$, $b = 10.418$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	90.238	252.098	90.238	217.077	90.238	217.077	90.238	217.077
$k(75)$	28.108	204.830	28.108	130.620	28.108	130.620	28.108	130.620
$k(80)$	0	160.086	0	43.920	0	43.920	0	43.920
$k(85)$	0	55.164	0	0	0	0	0	0

$\gamma = 2$ and $r^* = .0400$, $b = 9.331$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	141.989	278.158	140.964	277.339	126.466	276.157	126.466	273.886
$k(75)$	129.412	248.841	126.716	246.133	93.100	242.590	93.100	224.298
$k(80)$	113.011	213.648	107.827	207.629	50.294	200.228	50.294	158.179
$k(85)$	94.146	175.221	64.955	164.311	755	151.579	755	81.416

$\gamma = 5$ and $r^* = .0150$, $b = 10.418$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	125.347	250.209	110.585	251.670	110.585	253.175	110.585	243.060
$k(75)$	93.893	201.752	68.883	203.843	68.883	205.937	68.883	182.690
$k(80)$	46.387	156.458	25.692	158.192	25.692	159.751	25.691	119.867
$k(85)$	0	116.173	0	116.555	0	96.729	0	56.524

$\gamma = 5$ and $r^* = .0850$, $b = 8.487$

	Fair		$\varpi = .25$		$\varpi = .50$		$\varpi = .75$	
$k(65)$	150.000	300.000	150.000	300.000	150.000	300.000	150.000	300.000
$k(70)$	148.186	292.043	149.343	294.510	150.706	297.468	152.303	301.110
$k(75)$	141.085	274.429	143.274	279.176	145.916	285.033	149.068	292.449
$k(80)$	128.612	247.230	131.356	253.404	134.786	261.331	138.948	271.742
$k(85)$	111.575	212.180	113.986	218.151	117.198	226.342	121.134	237.668

ϖ annuity load charged to the consumer.

Figure 2.45: Consumption Trajectories. Case: $\{\gamma = .75,\ r^{*} = .0150,\ b = 10.418\}$

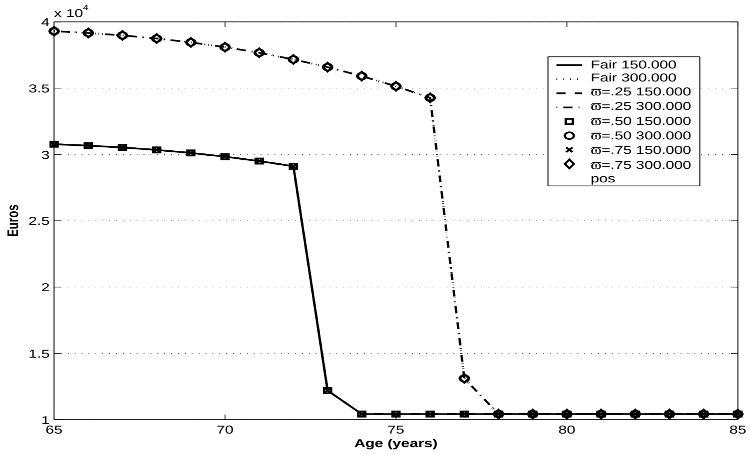


Figure 2.46: Consumption Trajectories. Case: $\{\gamma = 2,\ r^{*} = .0150,\ b = 10.418\}$

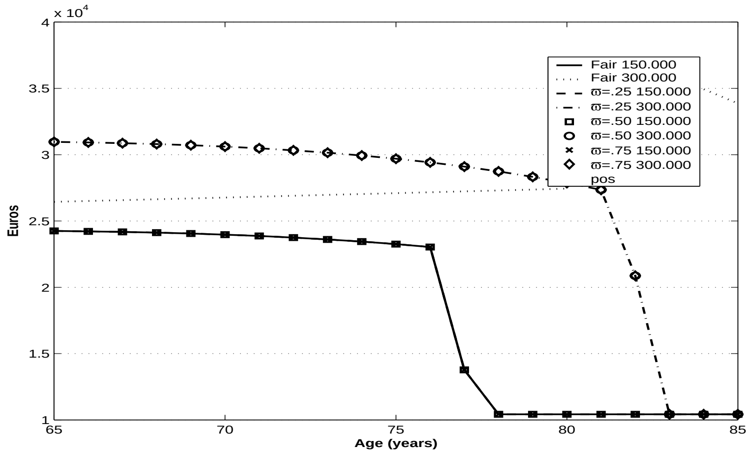


Figure 2.47: Consumption Trajectories. Case: $\{\gamma = 5,\ r^{*} = .0150,\ b = 10.418\}$

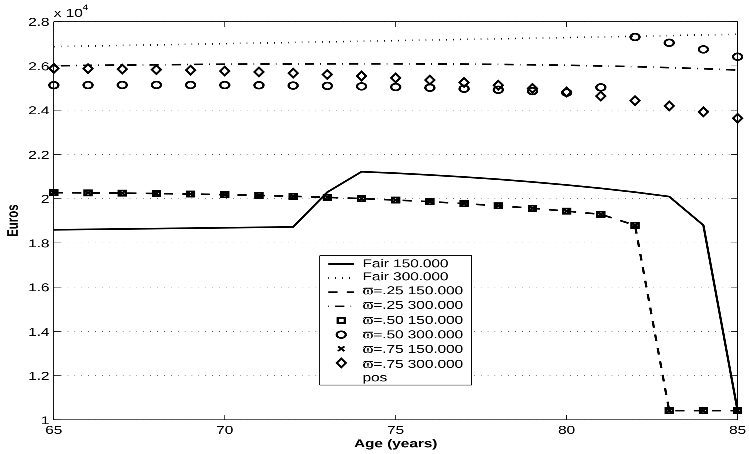


Figure 2.48: Consumption Trajectories. Case: $\{\gamma = .75,\, r^* = .0213,\, b = 10.034\}$

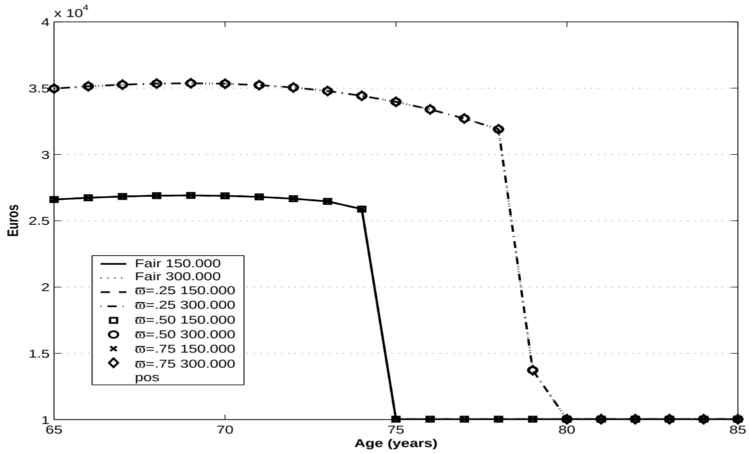


Figure 2.49: Consumption Trajectories. Case: $\{\gamma = 2,\, r^* = .0400,\, b = 9.331\}$

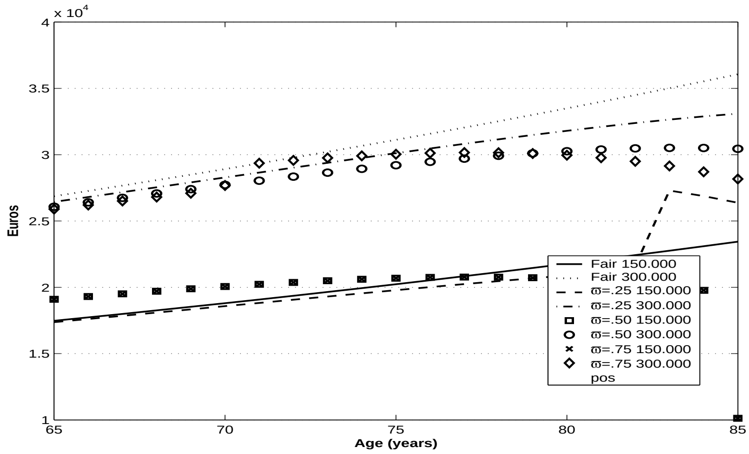
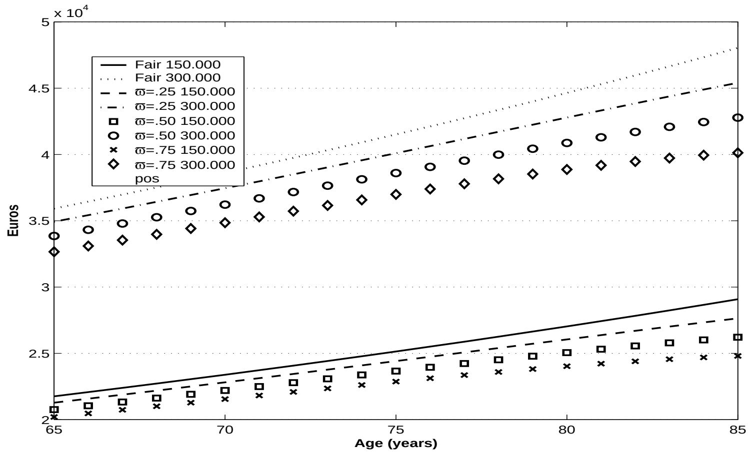


Figure 2.50: Consumption Trajectories. Case: $\{\gamma = 5,\, r^* = .0850,\, b = 8.487\}$



Chapter 3

Demand for Private Annuities and Social Security: Consequences to Individual Wealth

This chapter focuses on comparing public and private individual wealth over the life-cycle, when individuals face an uncertain length of life. We also analyze how a fully funded and actuarially fair Social Security affects the desire to annuitize private wealth. Within this framework, we find that a social security system can contribute to reaching a higher national wealth, even when the economy is composed of selfish individuals. Thus, by means of some simulations we obtain the result that a payroll tax of 6 percent increases individual wealth up to 17 percent. This increment, however, is obtained under the assumption that insurance companies offer fair annuities. On the contrary, under an unfair private annuity market, individual wealth can decrease around 10 percent for the same payroll tax.

3.1 Introduction

It has been well known since Feldstein (1974)[1] that Social Security crowds out private saving. The intensity of this crowding out effect varies according to how Social Security is financed, the behavior of each individual in the economy, and the return yielded by public and private pensions.

First, we know that the negative impact of a funded social security system on steady-state capital stock is smaller, or zero, than that yielded by an unfunded system. Thus for example, Auerbach and Kotlikoff (1987), İmrohoroğlu et al. (1999), and Conesa and Krueger (1999) estimate under an unfunded Social Security that steady-state capital stock is reduced between 11 and 25 percent. On the contrary, under a funded Social Security and lifetime uncertainty, Eckstein et al. (1985), Abel (1985), and Hubbard (1987) demonstrate that the crowding out only occurs when selfish individuals have neither access to the annuity market, nor actuarially fair annuities.

Second, individual feelings can influence the intensity of the crowding out effect as well. In particular, the more altruistic an agent is, the greater her saving and, therefore, her wealth is. Hence, Fuster (1999) finds that an unfunded social security system with two-sided altruistic agents crowds out only 8 percent of the capital stock for a 44 percent replacement rate. Note that this value is much lower than those estimated for selfish individuals by Auerbach and Kotlikoff (1987), among others. Nevertheless, there does not exist a consensus among economists regarding the importance of altruistic feelings on individual's behavior. Thus, we shall assume that our individual is selfish hereinafter.

Third, it has also been quantified that Social Security does not reduce the stock of capital in the long-run, so long as public and private pensions yield the same return. Unfortunately, this result has been obtained assuming that the decision of purchasing annuities is exogenous. As a consequence, we cannot derive any relationship between

[1]This negative effect was found firstly by Feldstein (1974) in the case of an unfunded Social Security.

the desire to purchase annuities and wealth over time.

In this chapter, we analyze how the wealth accumulation process is affected when both the Social Security is funded and individuals endogenously purchase annuities. To do so, we develop an economy that incorporates financial companies, private insurances, and a funded Social Security. Consequently, individuals can invest their wealth in safe assets, risky assets, and annuities. Moreover, in order to make the decision of purchasing annuities endogenous, we have made the following five assumptions: i) our individual faces an uncertain lifespan, ii) the yield of annuities dominates that of bonds, iii) a negative asset position at the time of death is forbidden, iv) the consumer is selfish, and v) she has a bounded rationality (i.e., even though financial institutions do not allow individuals to die in debt, our agent does not make decisions considering this constraint). Under the first four assumptions, Yaari (1965) states that the consumer will fully annuitize her savings. However, Sanchez-Romero (2005) demonstrates, by adding the assumption number v), that the decision of purchasing annuities depends on the relationship between the present value of future non-capital earnings and the initial wealth. That is to say, he finds that private annuities are not purchased when public benefits are high. Therefore, this last finding suggests that the crowding out effect should be analyzed not only by studying what sort of social security system the economy has, but also whether individuals are willing to purchase annuities or not.

On the other hand, the implications of these five assumptions are consistent with the fact that the demand for annuities is small on average. Nonetheless, there are other factors that explain the lack of annuitization, although they are out of the scope of this chapter. For example, bequest motive, annuity market imperfections such as the irreversibility of annuitization, or even risk sharing within families. The importance of any of these factors is, besides our assumption number v), that wealth accumulated at the age of retirement may change. This is in addition to the fact that wealth inequality, among descendants of people recently deceased, might increase over time.

Finally, it is worth noting that the utility function and the dynamic optimization method used throughout the chapter to calculate the optimal portfolio differ from previous analysis. Thus, instead of using a CRRA utility function and the Hamilton-Jacobi-Bellman method, as Merton (1971) and Richard (1975) have done before, we use a mean-variance utility and the Lagrange method, in order to be consistent with the bounded rationality assumption. Under this setting, we find two important features. One, the optimal portfolio is affected by age. Second, the investment in risky assets is much lower than those obtained by Merton (1971). Therefore, the investment in safe assets is preferred according to this model than in previous analyses.

Throughout the chapter we show that, when there is no Social Security or the payroll tax is equal to zero, our individual invests her wealth both in equities and in annuities. On the contrary, as the Social Security payroll tax increases, our agent is more willing to purchase bonds instead of annuities. According to this fact, we find that an actuarially fair funded social security system could increase the stock of capital in the long run if, and only if, our agent only purchases bonds at the beginning of her life-cycle. We simulate that the wealth increment, with a 6 percent payroll tax and private fair annuities, is close to 17 percent. However, under a private unfair annuity market this wealth increment is reduced, even to the point of decreasing wealth in the long run.

The remainder of this chapter proceeds as follows. Section 3.2 explains the consumer's behavior when there is no Social Security. Here, we obtain the optimal portfolio choice with the intention of subsequently estimating how wealth evolves over time. In section 3.3, we introduce a funded social security system. It will enable us to calculate how public and private wealth evolve according to different payroll taxes. Section 3.4 describes the effects of a funded social security system on the demand for private annuities. Furthermore, we shall distinguish between actuarially fair annuities and unfair annuities. In Section 3.5 we make our final conclusions. An Appendix containing a detailed demonstration of optimal investments under different theories as well as consumption behaviors is located throughout Sections 3.6 and 3.7. Some simulations finish the chapter.

3.2 Optimal Portfolio Choice under Uncertain Lifetime: Bonds, Equities, and Annuities

Individuals, who finance their future consumption using annuities, reduce the crowding out effect that is caused by an actuarially fair funded social security system, Abel (1986). Unfortunately, empirical research indicates that the value of the demand for annuities is small on average. Therefore, to substitute an unfunded Social Security by a funded one does not necessarily eliminate the crowding out effect.

A recent paper by Davidoff et al. (2005) suggests, among other reasons, that the lack of the demand for annuities may be caused by behavioral biases. Building on this idea, Sanchez-Romero (2005) proves that individuals with behavioral biases, such as bounded rationality, are more willing to purchase annuities the greater wealth is in relation to future non-capital earnings. Hence, *ceteris paribus*, we can expect that an actuarially fair funded Social Security causes a higher crowding out in economies with low private wealth.

The aim of this section, therefore, is to derive how individuals who live in an economy without Social Security accumulate assets to finance their future consumption at retirement. This result will be used as a benchmark to compare to the asset accumulation process derived by introducing a social security system. We develop an economy composed of financial companies which supply safe and risky assets (e.g. bonds and equities) and private insurances that offer annuities. The significance of the introduction of equities into the model is twofold. First, an economic model which studies private pensions needs to take into account how bonds and equities evolve. Second, if the agent has perfect foresight and short-selling is not constrained, then this model yields a greater accumulation of wealth which may lead to an increase in the demand for annuities. This point will be analyzed at the end of this section.

The representative economic agent faces an uncertain lifetime. Her survival probability Ω is known in advance, but the age that she will die is unknown. T is the maximum age to which the agent can survive. In addition, our economic agent has three key features which affect her investment decision making. First, the consumer is selfish. She does not leave an intentional bequest at death. Second, following Sanchez-Romero (2005), the agent does not take into account that financial institutions do not allow individuals to die in debt. So, we can say that our agent has a bounded rationality. This assumption affects the demand for annuities. For example, in order to anticipate consumption, individuals purchase annuities when they are young, and reject using annuities when they are retired. Third, the individual temporarily modifies her consumption according to financial markets expectations. Concretely, she increases her consumption while she expects to gain money investing in financial markets. This last assumption makes the consumption decision stochastic. Thus, instead of using an expected utility function, we use a mean-variance utility $v(c, \sigma_c^2)$, which satisfies the conditions demonstrated in Tsiang (1972). Therefore, the consumer's utility at age x is depicted by the following function U:

$$U(x) = \int_x^T \frac{\Omega(s)}{\Omega(x)} \beta(s-x) v\left(c(s,x), \sigma_c^2(s,x)\right) ds, \text{ for all } x \in [0,T). \tag{3.1}$$

Where $c(s,x)$ is the mean consumption at age s, of an x year old consumer, $\sigma_c^2(s,x)$ is the consumption variance at age s, of an x year old consumer. The function v is at least twice differentiable, strictly increasing in $c(s,x)$, and decreasing in $\sigma_c^2(s,x)$. $\frac{\Omega(s)}{\Omega(x)}$ is the probability that an individual of age x will be alive at age s, and $\beta(s-x)$ is the time discount factor from age x to age s, or $\mathrm{e}^{-\delta(s-x)}, \forall \delta \geq 0$.

Given a mean consumption level, the utility function (3.26) shows that the higher the consumption risk is, the lower the utility achieved by the consumer is. Hence, assuming that consumption variance is caused by risky asset investments, the consumer will maximize her consumption by investing in an efficient portfolio with the minimum variance and maximum expected return, as Sharpe (1964) and Markowitz (1952) suggest.

There are two alternative portfolios. The first one is composed by bonds and equities. The second one is composed by annuities and equities. Bonds and equities yield a safe interest rate r and a random interest rate α, respectively. Annuities, on the contrary, are lotteries contingent on the consumer mortality risk. Specifically, if the consumer survives at the end of the period, she will receive the safe interest rate r plus a risk premium μ contingent on her mortality risk. But, if she does not survive at the end of the period, she will not receive anything.

Each period, our representative individual has an initial wealth k and a labor income y. The individual takes $y(s), \forall s \in [0, T)$ as given. These resources are allocated to both consumption and investment. Nonetheless, she must choose the portfolio in which she will compound her resources. Thus, the agent at age x faces two alternative budget constraints.

$$k(x) + \int_x^T \frac{R(s)}{R(x)} \left((\alpha(s) - r(s))e(s,x) + y(s) - c(s,x)\right) ds = 0, \tag{3.2}$$

and

$$k(x) + \int_x^T \frac{R(s)}{R(x)} \frac{\Omega(s)}{\Omega(x)} \left((\alpha(s) - r(s))e(s,x) + y(s) - c(s,x)\right) ds = 0. \tag{3.3}$$

(3.2) and (3.3) are respectively the budget constraint when consumption is financed (besides by equities) by investing in conventional assets, and when consumption is financed by annuities. $e(s,x)$ is the amount of money invested in risky assets at age s, of an x year old consumer.[2] $\frac{R(s)}{R(x)}$ is the financial present value at age x, of a monetary unit received at age s, and $\frac{R(s)}{R(x)} \frac{\Omega(s)}{\Omega(x)}$ is the actuarial present value; that is,

$$\frac{R(s)}{R(x)} = \mathrm{e}^{-\int_x^s r(j)dj},$$

and

$$\frac{R(s)}{R(x)} \frac{\Omega(s)}{\Omega(x)} = \mathrm{e}^{-\int_x^s (r(j)+\mu(j))dj}.$$

It is worth noting that neither (3.2) nor (3.3) constrain wealth to be nonnegative along the lifespan. Nevertheless, the economic agent never dies in debt under (3.3), but she could under (3.2). This is an important property that we shall use subsequently. Also, if the consumer decides to purchase annuities, she will not leave a bequest. But, in contrast, if she chooses to finance consumption by investing in bonds, she will unintentionally bequeath at death. Therefore, choosing either (3.2) or

[2]Hereinafter, whenever the consumer will decide to purchase annuities, both mean consumption and money invested in risky assets will be denoted with a hat.

(3.3) has important consequences on income distribution inter and intra-generations. However, this fact is beyond the scope of this chapter.

So far we have established the general framework from which an individual accumulates assets to finance her future consumption. Now, we shall proceed by explaining the solutions obtained by plugging a CRRA utility function ($u(\xi) = \frac{\xi^{1-\gamma}}{1-\gamma}, \gamma > 0$) into (3.26), and assuming that consumption variance at age s is proportional to risky investment variance at age s, of an x year old consumer. That is,

$$\sigma_c^2(s,x) = \eta^2 \sigma_\alpha^2(s) e^2(s,x), \text{ for all } s, x \in [0,T) \text{ with } s > x, \tag{3.4}$$

where $\eta > 0$ is the constant of proportionality and σ_α^2 is the equity variance.

The agent maximizes (3.26) subject to either (3.2) or (3.3). Solving this economic problem yields two different consumption trajectories, which are quite similar to the uncertain lifetime case with just bonds and annuities.[3] Nevertheless, equities now modify the marginal utility of consumption and, consequently, the dynamic of consumption is also moved according to the expected evolution of asset returns. For example, consumption increases (resp. decreases) whenever the difference between asset returns also increases (resp. decreases). These consumption changes, nonetheless, are not high enough to produce consumption trajectories totally different from those obtained by Sanchez-Romero (2005). This circumstance is explained by the small investment in risky assets, depicted by any of the following equations:

$$e(x,x) = \frac{1}{\gamma}\left(\frac{\alpha(x) - r(x)}{\sigma_\alpha^2(x)}\right)\frac{\varphi(x,x)}{\eta^2}c(x,x),$$

or

$$\hat{e}(x,x) = \frac{1}{\gamma}\left(\frac{\alpha(x) - r(x) - \mu(x)}{\sigma_\alpha^2(x)}\right)\frac{\hat{\varphi}(x,x)}{\hat{\eta}^2}\hat{c}(x,x).$$

Where both φ and $\hat{\varphi}$ are functions whose range are restricted to the closed interval $[1,2]$. The first two components on the right side of the equality signal are similar to Merton (1971) and Richard (1975). However, the amount of money invested in risky assets depends on consumption, instead of depending on initial wealth and the

[3]The reader will find the analytical solutions in the Appendix. In order to compare these results with those obtained in Merton (1971) an additional Appendix has been included.

present value of future non-capital earnings. As a consequence, this model yields portfolios which are mainly composed of either bonds or annuities.[4] In particular, the proportion of either bonds or annuities relative to equities raises as our individual ages. Thus, equities are the main investment when the economic agent is young, but as time goes by she prefers to hold safer investments.

On the other hand, so long as Social Security does not pay benefits, wealth is also held in annuities rather than in bonds. Both the bounded rationality and the liquidity constraint assumptions are key factors for this allocation process. Thus, unless individuals purchase annuities or they have sufficient capital, they are unable to anticipate consumption at the beginning of their life cycle. Therefore, we find that young individuals are more willing to purchase annuities in order to increase their consumption. However, the presence of annuities raises borrowed money and so, because individuals must repay their debts, the economic agents have a lower positive asset position upon retirement. This latter fact negatively affects the demand for annuities, Sanchez-Romero (2005). Nonetheless, they will buy insurances contingent on their death due to the lack of public benefits assumed so far.

In sum, in an economy without Social Security, we find that our agent allocates her wealth in a portfolio composed by equities and annuities. But, equities represent a small percentage of total wealth, and annuities decrease wealth held upon retirement among those individuals who have needed to borrow money at young ages.

[4]According to Tsiang (1972), the CRRA utility function $u(y)$ is convergent to $v(c, \sigma_c^2)$ if, and only if:

$$\eta \geq \left(\frac{\varepsilon(1+\gamma)}{2} + \frac{1}{\varepsilon\gamma} \right) \cdot \max \left\| \frac{\alpha(x) - r(x)}{\sigma_\alpha(x)} \right\|_{\forall x \in [0,T)},$$

where ε is a real number which satisfies that $\frac{\sigma_c}{c} \leq \varepsilon < 1$. In particular, Tsiang (1972) suggests a value of $\frac{1}{10}$ for ϵ, therefore we cannot expect high values of $\frac{\varphi(x,x)}{\eta^2}$. Note that this condition corresponds to the non-annuitized wealth case. Thus, if we are interested in the value of $\hat{\eta}$, we should add the mortality risk premium to $r(x)$.

3.3 Payroll Tax and Wealth-Age Profiles

Up to now, we have studied the asset accumulation process of an individual who lives in an economy without Social Security. Under this scenario, we have found that individuals mainly purchase annuities because it enables one to borrow money, and because it assures an income after retirement. In this section however we introduce an actuarially fair funded social security system that assures an income at retirement. Thus, Social Security levies a payroll tax τ on gross earnings, in exchange of a future benefit when people retire. According to this fact, we rewrite income as the following piecewise function:

$$y(s) = \begin{cases} (1-\tau_e)w(s) & 0 \leq s < J \\ b(s) & s \geq J \end{cases}, \tag{3.5}$$

where $w(s)$ is the gross salary at age s, $b(s) = b$, for all s, is the flat public pension benefit received at retirement, and J is the age of retirement. We consider that the payroll tax is paid not only by the employee τ_e, but also by the employer τ_f. As a consequence, our representative individual receives an actuarially fair pension benefit equal to:

$$b = (\tau_e + \tau_f)\frac{\int_0^J R(s)\Omega(s)w(s)ds}{\int_J^T R(s)\Omega(s)ds}. \tag{3.6}$$

This assumption is introduced into the model because current social security regimes are jointly financed by employers and employees. In addition, the fact that Social Security is financed by these two agents has important and interesting consequences on individual saving. For example, a funded social security system financed by employers and employees generates an increase in lifetime resources. The positive income effect caused by the system, however, differs according to the portfolio chosen by each individual. Thus, in a model without firms, Hubbard (1987) proves that an actuarially fair and funded system generates an increase in lifetime resources when individuals do not purchase actuarially fair annuities. Nevertheless, a system partially financed by employers generates an increase in lifetime resources as well, even when individuals purchase annuities. That is to say, substituting equation (3.5) and (3.6) into the budget constraint (3.3), and afterwards subtracting (3.3) with respect to the budget constraint without Social Security, we have that

$$\tau_f \int_0^J R(s)\Omega(s)w(s)ds. \tag{3.7}$$

This increment in resources correspond to those individuals who purchase actuarially fair annuities. (3.7) equals the pension financed by the employer; since we are assuming that $w(s)$ is the maximum gross salary, that the employer is willing to pay without Social Security. On the contrary, if our individual decides to finance her consumption with the portfolio composed by bonds, we will expect a greater increment in lifetime resources than if it is financed by annuities.[5] Repeating the previous process, but now with the budget constraint (3.2), we get that

$$\kappa \left(\tau_f \int_0^J R(s)\Omega(s)w(s)ds + \tau_e \int_0^J R(s) \left(\Omega(s) - \frac{1}{\kappa} \right) w(s)ds \right), \tag{3.8}$$

where κ is the difference in discount rates under certainty and uncertainty:

$$\kappa = \int_J^T R(s)ds \Big/ \int_J^T R(s)\Omega(s)ds > 1.$$

We have found according to (3.7) and (3.8) that an actuarially fair funded Social Security could raise lifetime resources. On the one hand, we know that the higher the income effect is, the greater the payroll tax is. On the other hand, the income effect also increases when our individual decides to invest in bonds, instead of doing so in annuities. Consequently, given a periodical earning such as (3.5), we can enumerate three causes that reduce private saving: i) a decrease in net salary, ii) an increase in consumption due to the positive income effect, and iii) a lower necessity of saving for retirement motive. Nevertheless, the decrease in private savings is offset by an increase in public savings. Therefore, it is not clear that the individual wealth[6] will be reduced in the long run. In fact, individual wealth may either increase or decrease depending on how the payroll tax modifies both public and private wealth over time. In particular, we expect that Social Security will raise (resp. reduce) individual wealth accumulated, so long as the elasticity of public savings with respect to the

[5] Given that κ also depends on Ω, we expect that $\Omega(s) > \frac{1}{\kappa}$ for almost all ages (s) between x and J years old.

[6] Hereinafter we call "individual wealth" as the sum of private and public wealth.

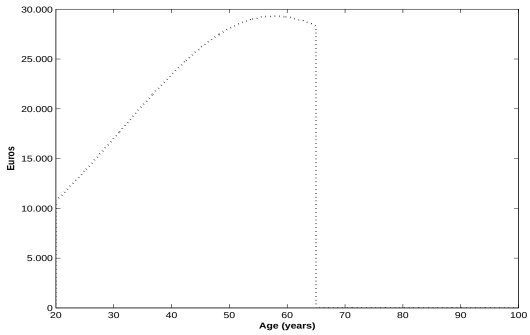
payroll tax is greater (resp. lower) than the absolute value of the elasticity of private savings with respect to the payroll tax.

In order to understand how individual wealth evolves over time, we simulate nine wealth profiles which differ according to the payroll tax and the proportion of the tax levied by each economic agent. To do so, we assume that bonds yield an annual constant interest rate r equal to 0.037. Equities yield an interest rate that follows an Ito process

$$\alpha(s)ds = r(s)ds + \sigma_\alpha dB(s),\ dB(s) \sim N(0, \sqrt{ds})$$

where σ_α equals 0.1. The individual satisfies every feature explained in section 3.2, with a γ-value of 2, and a time discount factor δ of 0.02. Moreover, we assume that the gross earning received by the individual, which is used to calculate these wealth profiles, is depicted by Figure 3.1 below.

Figure 3.1: Gross Earning Profile ($w(s)$)



The actuarially fair funded Social Security offers an implied rate of return equal to the mortality hazard rate plus bonds return. Table 3.1 shows the annual pension benefits that our individual will receive for different payroll taxes (i.e. $\tau = \tau_e + \tau_f$). On the one hand, it shows that a total payroll tax of 3 percent roughly assure a benefit equal to the lowest income of her life. On the other hand, values of 6 and 8 percent points approximately guarantee 85 percent of her average earning and her highest earning, respectively. We have chosen these percentages because they are the most important three cases, which will be explained subsequently. In addition, a percentage greater than 8 percent makes no sense because it has perverse effects both on private saving and on the economy.

Table 3.1: BENEFITS (b)

Payroll Tax (τ)	Annual Pension Benefit
0,03	8.727,70
0,06	17.455,00
0,08	23.274,00

Note: The individual retires at the age of 65. The mortality hazard rate is assumed to follow the Gompertz's Law $\mu(s) = \alpha e^{\beta s}$, where α is equal to $9,221765 \cdot 10^{-5}$ and $\beta = 0,085277$.

Given this setup, Figure 3.2 shows that our individual borrows money at the beginning of her life-cycle in order to anticipate her consumption. However, the money borrowed decreases as the payroll tax increases (dotted square line). Note in Figure 3.2 that changing the total payroll tax τ from 3 to 6 raises individual wealth. By contrast, Figure 3.3 shows that a payroll tax of 8 percent leads our individual to not save for retirement (dotted line with an x mark); as a consequence total wealth is almost the same as an economy without Social Security (solid line).

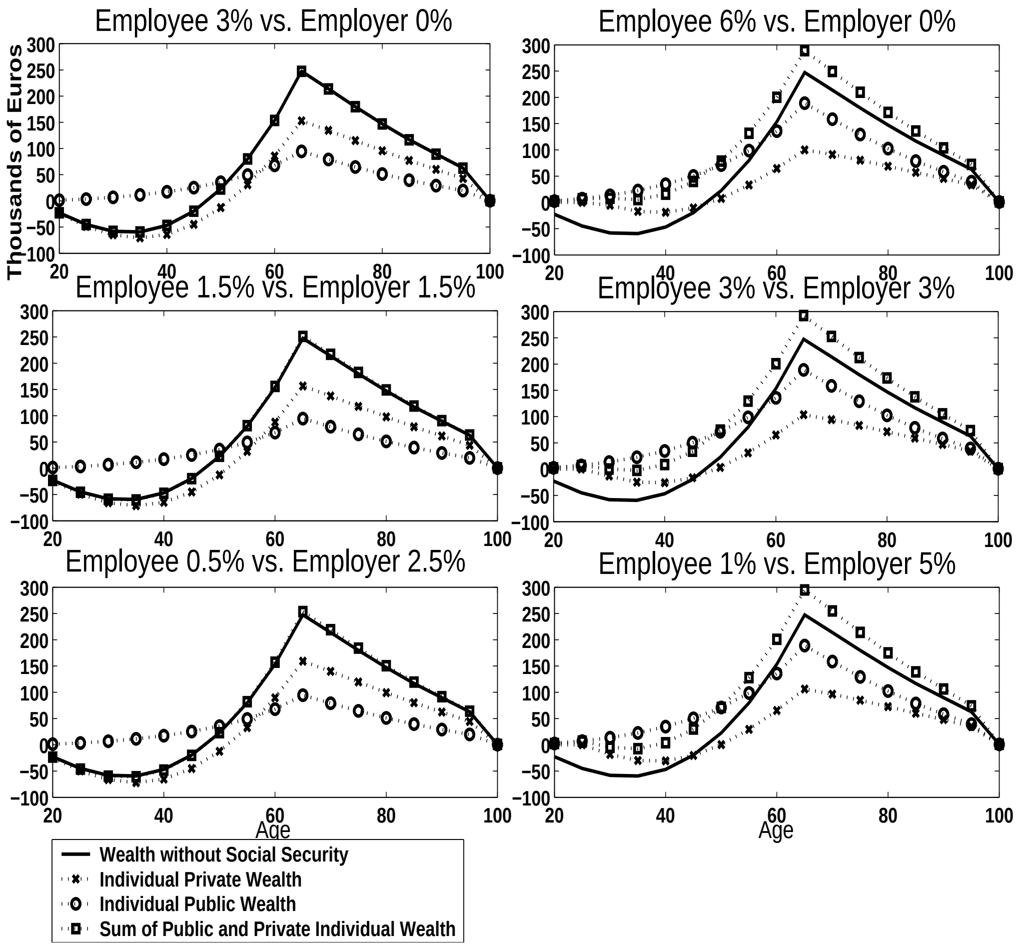
We have found that the increment of total wealth occurs because young individuals are not willing to purchase annuities. However, if our individual does not purchase annuities along her lifespan, as happens in Figure 3.3, there will not be such an increment. Hence, there must exist a payroll tax that maximizes individual wealth without strangling private savings. In this particular case the optimal payroll tax is equal to 6, as Table 3.2 shows.

Table 3.2: INDIVIDUAL WEALTH AT THE AGE OF 65

τ_e ($\tau_f = 0$)	0	1	2	3	4	5	6	7	8
Percentage	100	100	100	100	100	105	117	89	102

Note: $100 = 247.519,26$ euros.

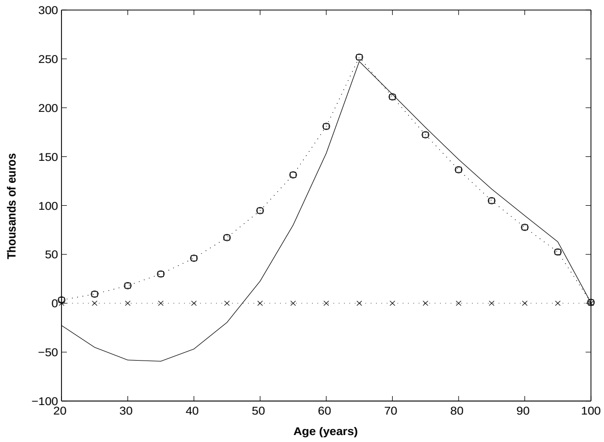
Figure 3.2: Wealth Profiles: Payroll Taxes 3 and 6 Percents



This fact implies that even though a social security system leads young individuals to be worse off in terms of consumption, their wealth become greater as they age (if, and only if, the system has not excessively levied gross earnings). Thus, the system does not necessarily offset one public monetary unit by another private one. In fact, a different payroll tax can help to raise individual wealth.[7] Another alternative for raising individual wealth is to increase τ_f and decrease τ_e. However, we have not found significant changes to wealth, as Figure 3.2 shows, that could balance the negative effect on unemployment caused by the increment on the labor cost versus the capital cost.

[7]Note that Table 3.2 is calculated under the assumption that τ_f equals 0.

Figure 3.3: Wealth Profile: Payroll Tax 8 Percent



Note: in this case, there is no difference between the proportion of the tax paid by each agent.

In sum, an actuarially fair funded Social Security expels the demand for private annuities, but it may increase individual wealth as well. The former effect has been widely discussed since Feldstein (1974). By contrast, the latter effect results if, and only if, the following two circumstances take place: i) individuals voluntarily decide not to purchase annuities and ii) financial markets do not allow individuals to die in debt. Therefore, this result shows, contrary to previous research, that a social security system can contribute to reaching a higher national wealth, even when the economy is made up of selfish individuals.

3.4 Effects of a Fully Funded Social Security on the Demand for Private Annuities

It has been pointed out that an actuarially fair and fully funded social security system does not reduce the steady-state wealth whenever individuals are selfish and a private annuity market exists. In order to obtain this result, it is necessary to assume that individuals are rational. Otherwise, if individuals have a bounded rationality of the sort explained in this chapter, the actuarially fair and fully funded Social Security can either increase or decrease steady-state wealth (see Table 3.2). Since the introduction of the system reduces the desire of purchasing annuities and, as a consequence, individuals can either have a greater individual wealth because they do not borrow money at young ages, or have a lower individual wealth because they do not save for retirement. Therefore, we analyze in this section the possible reasons for not investing in private annuities and how it affects individual wealth.

The introduction of this social security system yields two reasons for not investing in annuities. First, it causes a lower private wealth upon retirement[8] that reduces the desire of purchasing annuities. As it is explained by Sanchez-Romero (2005). Second, following Hubbard (1987), individuals may prefer bonds to annuities in order to achieve higher lifetime resources, see (3.8). Thereby, the higher the contribution to Social Security is, the greater the crowding out effect on the demand for private annuities is. However, the first reason is offset because we have assumed that financial institutions do not allow individuals to die in debt. Thus, Social Security may increase private wealth by inducing individuals to hold their wealth in the form of bonds; since once they purchase bonds instead of annuities, they are unable to borrow money and so they have a greater positive asset position earlier.[9] The intensity of these two opposite effects on private wealth is the key factor to determine whether or not the system produces a crowding out. In particular, we find that Social Security raises wealth while it does not cancel private saving for retirement.

This current section proceeds as follows. First, we explain the demand for private annuities when there is no Social Security. We use its annuity equivalent wealth values (AEW) as our baseline case. Subsequently, we divide this section in two subsections in order to give insight into how Social Security changes the demand for private annuities. One subsection shows an individual's behavior when private markets offer fair annuities, and the other subsection shows the individual's behavior when they offer unfair annuities. Both subsections contain tables and figures which depict the desire to purchase annuities for different payroll taxes and risk aversion

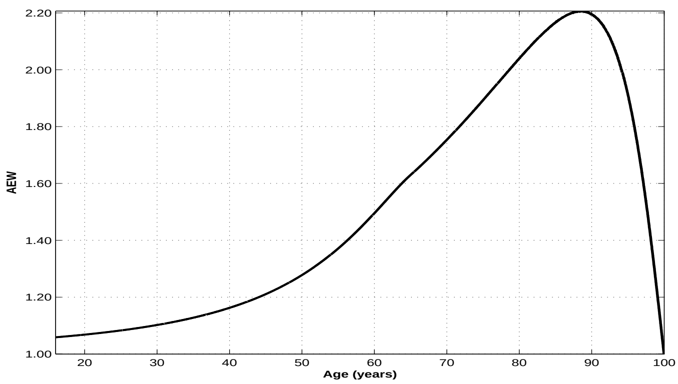
[8] This is equivalent to say that Social Security reduces private saving for retirement.

[9] In order to realize this fact, compare in Figure 3.2 those charts on the left side with those on the right side.

coefficients.

We found in section 3.3 that young people prefer annuities to bonds in order to anticipate consumption. This is because financial institutions do not lend money unless people insure their wealth with life insurances. Later on, assuming an economy without public pensions, individuals prefer to purchase annuities in order to maintain their economic status. If they choose, by contrast, the alternative portfolio composed of bonds, then they have the risk of outliving their financial resources more quickly. Equivalently, in the case of holding their wealth in bonds, individuals may not have an income in the time just before death. Therefore, people always prefer to purchase annuities when there is no Social Security. Figure 3.4 below shows this statement for the representative agent introduced in the previous section. Note that AEW values[10] are higher than one, and thus annuities are preferred over bonds. AEW has a Λ-shape which means that this individual is more willing to purchase annuities as she approaches the date of retirement; while she is almost indifferent when choosing between bonds and annuities both at the beginning of her life-cycle and at the end.

Figure 3.4: A.E.W. by Age Without Social Security



Note: An annuity equivalent wealth value lower (resp. greater)than one means that the individual prefers (resp. does not prefer) bonds to annuities.

[10]The proportion of annuitized wealth that is necessary to achieve the utility level when the consumer has no access to the annuity market.

The introduction of Social Security will move the AEW figure downwards. Thus, given that AEW has a Λ-shape, we have to expect that the system mainly affects our individual when young, conditioning her future decisions afterwards. In addition to age, Figure 3.4 also changes according to the behavior towards risk and the proportion of the load charged upon annuities.

3.4.1 *Perfect Life Insurance*

Private annuity markets, which offer actuarially fair life insurances, assure that individuals' lifetime resources raise according to either (3.7) or (3.8). Consequently, every result already obtained is applicable. Here, we focus on studying how private wealth is modified by different payroll taxes and risk aversion coefficients. This is because, following Sanchez-Romero (2005), the demand for private annuities mainly depends on private wealth and on future benefits. In order to analyze this fact, we will first pay attention to our agent at the age of 65, see Table 3.3 below. Second, we shall study with the help of Figure 3.5 the demand for private annuities in a dynamic perspective.

Table 3.3: Private Wealth at the age of 65

Payroll Tax	Risk Aversion Coefficient		
τ_e vs. τ_f	$\gamma = 0.75$	$\gamma = 2$	$\gamma = 5$
3-0	292.550,10[a]	153.032,06[a]	114.137,23[a]
1,5-1,5	298.355,56[a]	156.744,85[a]	117.266,63[a]
0,5-2,5	302.225,87[a]	159.220,04[a]	119.352,89[a]
6-0	213.502,81[a]	100.001,84[a]	55.353,57[a]
3-3	222.280,32[a]	103.725,10[a]	58.056,57[a]
1-5	227.218,18[a]	106.177,29[a]	59.878,10[a]
8-0	0,00[b]	0,00[b]	0,00[b]
4-4	0,00[b]	0,00[b]	45.476,75[a]
1,33-6,67	0,00[b]	0,00[b]	49.068,69[a]

[a] The individual decides to annuitize her private wealth.

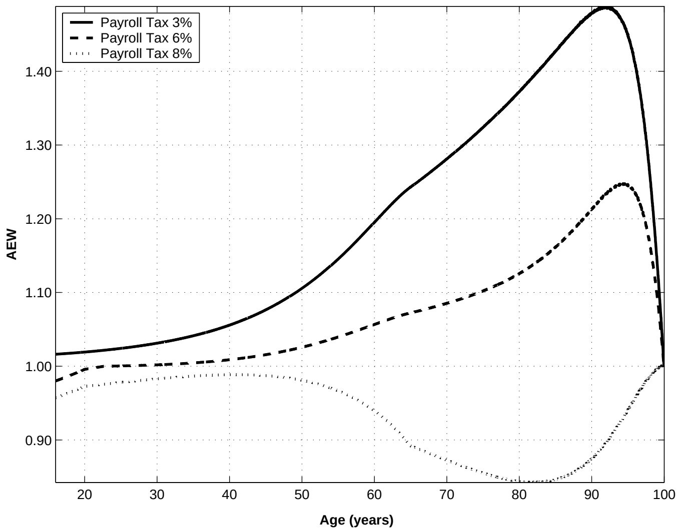
[b] The individual prefers to hold her private wealth in bonds.

Table 3.3 above shows whether our individual purchases annuities (superscript a) or not (superscript b) according to her private wealth and her risk aversion coefficient. Thus, the table contains three important features: i) given the gross earning profile of Figure 3.1 and using Table 3.1, we find that our individual purchases annuities so long as the payroll tax is lower than 8 percent. This is an important result not only because she achieves, according to Table 3.2, a greater wealth, but also because it assures a periodical income up to her death. ii) it is worth noting that in this model the risk aversion coefficient causes two opposite effects upon the demand for private annuities. On the one hand, it is well known that the higher the risk aversion coefficient is, the greater the desire of an agent to purchase annuities is. However, on the other hand, we see in Table 3.3 that the lower the γ value is, the greater the private wealth at the age of retirement is. Thus, the agent is more willing to purchase annuities. In sum, once again the risk aversion coefficient does not explain the demand for annuities. Finally, iii) private wealth increases as the proportion of the payroll tax paid by the employer increases. This wealth increment nonetheless is not high enough to balance the resources paid by the employer[11] except for the case in which both the individual is quite risk adverse and the payroll tax is greater or equal than 8 percent.

In addition to the static analysis presented in Table 3.3, Figure 3.5 above shows the AEW values by age associated with the following payroll taxes: (3-0), (6-0) and (8-0). The solid line plots how our individual always prefers annuities to bonds with a payroll tax of 3 percent. A payroll tax of 6 percent (dotted line) causes our individual to decide to purchase bonds instead of annuities at the beginning of her life-cycle. The dashed line plots how she always purchases bonds with a payroll tax of 8 percent. Therefore, AEW values by age are pushed downwards as the payroll tax increases (in order to see how AEW by age evolves, compare Figure 3.5 with Figure 3.4). However, once the individual has decided to purchase bonds along the rest of her life, the AEW by age has an inverted Λ-shape. As a consequence, we

[11] According to equation (3.6) the amount of the benefit received only depends on the total payroll tax, i.e. $\tau_e + \tau_f$.

Figure 3.5: A.E.W. by Age With Social Security and Fair Annuities



do not expect that she will reject her investment once she has decided the asset in which she allocates her wealth. In fact, as seen in Figure 3.5, financial institutions will have to offer greater returns in order to be able to make people change from one asset to another.[12]

3.4.2 *Imperfect Life Insurance*

In the real world we do not find actuarially fair annuities. In general, annuities are loaded by insurers with the intention of financing reserves, administrative costs, commissions, and profits. Therefore, it is more realistic to analyze previous results when life insurances do not offer fair annuities. The first consequence of this fact is that an actuarially fair funded Social Security offers a higher rate of return than private annuities, and hence individuals achieve a greater wealth by investing in public pensions than in private annuities. Second, an imperfect annuity market cannot offset those annuities offered by the social security system. This situation causes both an income effect and a substitution effect that change the demand for private annuities when fair life insurances were offered. Specifically, a lower annuity

[12]Thus, if policy makers aim to annuitize private pension plans, then it is convenient to undertake policies when people are between 30 and 50 years old.

return increases present consumption and diminishes future consumption due to the substitution effect. Thus, our individual either consumes all her income if she invests in bonds, or borrows more money at the beginning of her life-cycle, and subsequently increases her saving, in the case of investing in annuities. On the other hand, given that public benefits are actuarially fair, an imperfect private annuity market reduces the income effect produced by investing in bonds. In order to show this fact, we assume for the sake of simplicity that annuities yield the following rate of return at age s:[13]

$$r(s) + (1 - \varpi)\mu(s), \text{ for all } s \in [x, T),$$

where $\varpi \in (0, 1)$ is the percentage of load over the mortality hazard rate. Note that we use this formula in order to satisfy that the yield of annuities still dominates that of bonds. Thereby, (3.7) converges to (3.8) as we give to ϖ a value close to 1. Thus (3.7) is now rewritten as

$$\hat{\kappa}\left(\tau_f \int_0^J R(s)\Omega(s)w(s)ds + \tau_e \int_0^J R(s)\left(\Omega(s) - \frac{\hat{\Omega}(s)}{\hat{\kappa}}\right)w(s)ds\right), \tag{3.9}$$

where $\hat{\kappa}$ is the difference in discount rates under unfair annuities and fair ones:

$$\hat{\kappa} = \int_J^T R(s)\hat{\Omega}(s)ds \bigg/ \int_J^T R(s)\Omega(s)ds > 1.$$

From (3.9) we derive, whenever insurers offer unfair annuities, that the positive income effect caused by switching from annuities to bonds is diminished. According to this effect, annuities are now more preferred than bonds. However, the latter cannot balance the substitution effect. Indeed, we can see comparing Tables 3.3 and 3.4 below, that bonds are now more preferred than annuities at the age of 65.

[13]Now, Ω has been transformed to $\hat{\Omega}$ which has the following formula:

$$\hat{\Omega}(x) = \mathrm{e}^{-(1-\varpi)\int_0^x \mu(j)dj} > \Omega(x), \text{ for all } x \in [0, T).$$

Table 3.4: PRIVATE WEALTH UNDER UNFAIR ANNUITIES AT THE AGE OF 65

Payroll Tax τ_e	Load ϖ	Risk Aversion Coefficient $\gamma = 0.75$	$\gamma = 2$	$\gamma = 5$
3	0,25	245.527,90[a]	148.750,39[a]	119.05753[a]
	0,50	199.331,52[a]	142.438,29[a]	124.514,48[a]
	0,75	141.062,91[b]	131.942,28[a]	130.045,47[a]
6	0,25	0,00[b]	100.890,77[a]	57.383,95[a]
	0,50	0,00[b]	33.473,84[b]	60.787,67[a]
	0,75	0,00[b]	33.473,84[b]	66.450,95[a]
8	0,25	0,00[b]	0,00[b]	0,00[b]
	0,50	0,00[b]	0,00[b]	0,00[b]
	0,75	0,00[b]	0,00[b]	0,00[b]

[a] The individual decides to annuitize her private wealth.

[b] The individual prefers to hold her private wealth in bonds.

We realize in Table 3.4 that private wealth decreases more markedly as the risk aversion lowers (see columns with $\gamma = 2$ and $\gamma = 0,75$) and the load increases. Instead, a γ value equal to 5 yields a higher private wealth under unfair annuities than under fair ones. This is so because she prefers bonds to annuities at the beginning of her life-cycle and, as a consequence, she cannot borrow money because she simply consumes her income during this period. Moreover, we have used three different loads {0,25; 0,5; 0,75} with the aim of showing how the demand for private annuities mainly depends on the relationship between private wealth and the present value of future earnings. Thus, it is worth noting that any of these loads yield, by definition, an annuity internal rate of return greater than that of bonds ($r = 0,037$); in particular, at the age of 65 they are equal to {0,049; 0,046; 0,042} respectively. Therefore, the more unfair annuities are, the greater the present value of future benefits with respect to current private wealth is. Thus, the individual is less willing to purchase annuities.[14] In addition to the relationship between private wealth and future earnings, the risk aversion coefficient γ has to be considered as well,

[14] Read proposition 2.1 in Chapter 2.

given that it determines the threshold private wealth from which our individual switches her investments from annuities to bonds. For example, Table 3.4 shows that, when annuities are not fair, an individual with both a γ equal to $0,75$ and an annual pension benefit of 17.455 euros[15] decides not to invest either in bonds, nor in annuities, for retirement. By contrast, in the subsection 3.4.1, Table 3.3 shows that under the same features our individual accumulates $213.502,81$ euros by investing in fair annuities. Thus, we can note that the threshold private wealth is easily reached, so long as the risk aversion decreases and the load increases.

Table 3.5: Individual Wealth at the age of 65 under Unfair Annuities ($\gamma = 2$)

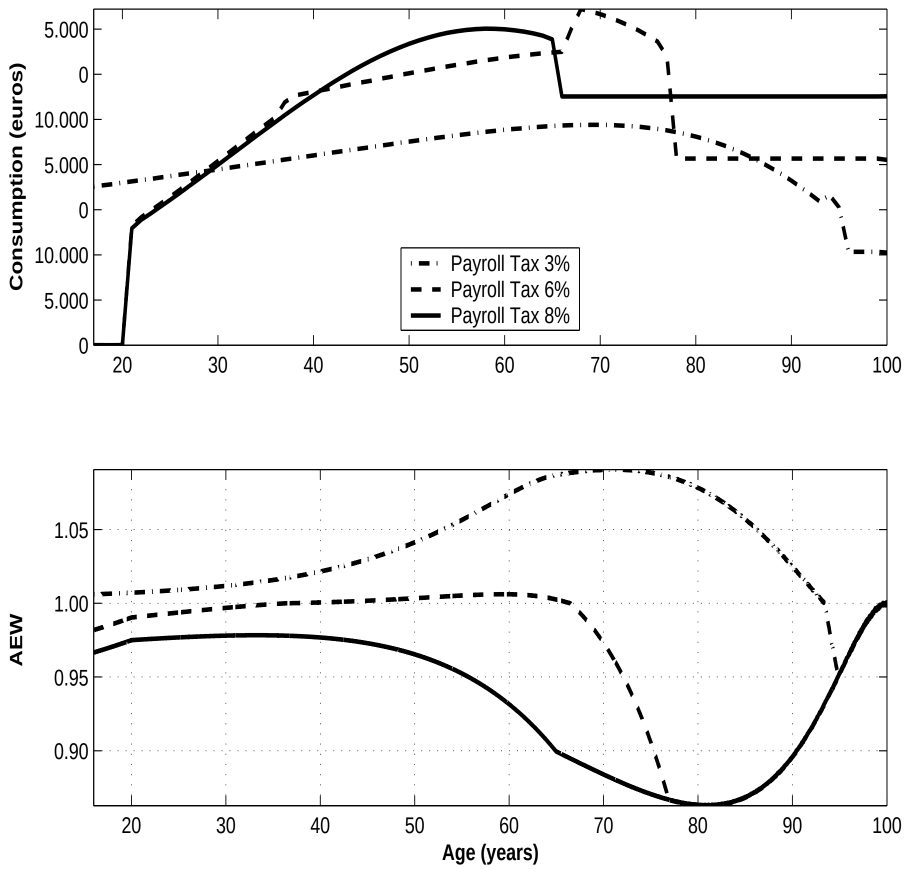
τ_e ($\tau_f = 0$)	0	1	2	3	4	5	6	7	8
$\varpi = 0,25$	102	101	99	98	97	106	117	89	102
$\varpi = 0,50$	104	101	99	96	95	107	90	89	102
$\varpi = 0,75$	107	102	97	91	97	94	90	89	102

We have used as benchmark $100 = 247.519,26$ euros, which corresponds to the individual wealth achieved under fair annuities and no Social Security (see Table 3.2). Note that individual wealth is greater than our benchmark case for payroll taxes 1, 5 and 6. Nonetheless, the difference is reduced, and is even negative, as the load approaches to one.

In sum, assuming an actuarially fair funded Social Security, individual wealth at the age of retirement is negatively affected by an unfair annuity market. This is so, unless policy makers decide to either reduce the payroll tax below 3 percent, or increase it up to 8 percent (see Table 3.5). However, if we take the first decision, we expect that people will outlive their financial resources faster and, consequently, their consumption will decrease as time goes by. Therefore, looking at consumption trajectories depicted in Figure 3.6 below, we recommend levying a payroll tax of 8 percent, not only because it assures an income after retirement, but also because individual wealth is not depleted before death.

[15] See Table 3.1.

Figure 3.6: Consumption, Individual Wealth and A.E.W. by Age, With Social Security and Unfair Private Annuities (ϖ=0,50; $\gamma = 2$)



Note: annuities are only purchased when the payroll tax is lower than 3 percent.

3.5 Concluding Remarks

This chapter presents new results about the crowding out effect produced by an actuarially fair funded Social Security on the stock of capital. We find that our consumer is more willing to purchase bonds, instead of annuities, as the payroll tax levied increases. On the one side, Social Security diminishes private wealth upon retirement which reduces the desire of purchasing annuities. On the other side, our individual may prefer bonds to annuities in order to achieve higher lifetime resources. We also find that, although this social security system expels the demand for private annuities, it may increase individual wealth. This latter fact nonetheless only happens so long as our individual voluntarily decides not to purchase annuities at the beginning of her life-cycle and, furthermore, that financial markets do not allow individuals to die in debt.

These findings show, contrary to previous research, that a social security system can contribute to reach a higher national wealth, even when the economy is composed by selfish individuals. For example, some simulation exercises presented here point out that a payroll tax of 6 percent increases individual wealth up to 17 percent points. This increment however is obtained under the assumption that private insurers offer fair annuities. Thus, on the contrary, under an unfair private annuity market, individual wealth can decrease around a 10 percent for the same payroll tax.

The importance of these findings raise some questions for future research. The most important is to determine the optimal payroll tax under an unfunded Social Security. Since, given the increasingly concern in developed countries about the feasibility of the social security system, a similar finding, as the one presented here, could contribute not only to decrease the payroll tax for future generations of workers, but also to give new reasons for maintaining the current social security system.

3.6 Appendix

Our agent decides each time whether to annuitize her wealth or not. This circumstance lies on the assumption (v) (bounded rationality) introduced in this model. As a consequence, our individual compares the utility reported by annuitizing her wealth with not doing so. Thus, we maximize her expected utility twice regarding either equation (3.2) or equation (3.3). But, because the algebra in both processes are similar, we shall only derive the optimal consumption and investment at age x, when our individual decides not to annuitize her wealth.

Optimal Consumption and Investment at age x under a Non-annuitized

Wealth.

Assuming that our agent at age x maximizes equation (3.26), subject to (3.2) and (3.4) then, we can compute the optimal allocation process as an isoperimetric problem, whose equation is

$$\begin{aligned}\Im \equiv \Im(\mathbf{c}, \mathbf{e}, \lambda(x)) = \int_x^T \tfrac{\Omega(s)}{\Omega(x)} \beta(s-x) \left(\tfrac{c(s,x)^{1-\gamma}}{1-\gamma} - \tfrac{\gamma}{2} \tfrac{\sigma_\alpha^2(s)\eta^2 \mathrm{e}^2(s,x)}{c(s,x)^{1+\gamma}} \right) ds \\ + \lambda(x) \left(k(x) + \int_x^T \tfrac{R(s)}{R(x)} \left(\theta(s)\sigma_\alpha(s)\mathrm{e}(s,x) + y(s) - c(s,x) \right) ds \right)\end{aligned}$$

where $\theta(s) = \frac{\alpha(s)-r(s)}{\sigma_\alpha(s)}$.

The first-order conditions at age x for $\mathbf{c}$, $\mathbf{e}$ and $\lambda(x)$, respectively, are

$$c(x,x)^{-\gamma} + \frac{\gamma(1+\gamma)}{2} \sigma_\alpha^2(x)\eta^2 \mathrm{e}^2(x,x) c(x,x)^{-2-\gamma} - \lambda(x) = 0, \tag{3.10}$$

$$-\gamma\sigma_\alpha^2(x)\eta^2 \mathrm{e}(x,x) c(x,x)^{-1-\gamma} + \lambda(x)\theta(x)\sigma_\alpha(x) = 0, \tag{3.11}$$

$$k(x) + \int_x^T \frac{R(s)}{R(x)} \left(\theta(s)\sigma_\alpha(s)\mathrm{e}(s,x) + y(s) - c(s,x) \right) ds = 0. \tag{3.12}$$

Now, we should follow the next six steps in order to derive $c(x,x)$ and $e(x,x)$. Firstly, we derive the function $e(x,x)$ from (3.11). Second, we plug $e(x,x)$ into (3.10) and multiply both sides of the equation by $c(x,x)^\gamma$. Third, let define the function

$$\varphi(s,x) = \frac{\lambda(x)}{\beta(s-x)} \frac{R(s)}{R(x)} \frac{\Omega(x)}{\Omega(s)} c(s,x)^\gamma, \forall s \in [x,T) \tag{3.13}$$

and introduce it into the last equation. Thus, by solving the second-order equation in the variable $\varphi(s,x)$, it is easy to prove that $\Im$ is maximized if, and only if:

$$\varphi(s,x) = \frac{1 - \sqrt{1 - 2\frac{1+\gamma}{\gamma} \left(\frac{\theta(s)}{\eta} \right)^2}}{\frac{1+\gamma}{\gamma} \left(\frac{\theta(s)}{\eta} \right)^2} \text{ for all } x \in [0,T).$$

Fourth, using (3.13) and $\varphi(s,x)$, we obtain that $c(s,x)$ and $e(s,x)$ are

$$c(s,x) = \left(\frac{1}{\lambda(x)} \right)^{\frac{1}{\gamma}} \psi_x(s), \tag{3.14}$$

$$e(s,x) = \frac{1}{\gamma} \left(\frac{\alpha(s)-r(s)}{\sigma_\alpha^2(s)} \right) \frac{\varphi(s,x)c(s,x)}{\eta^2}, \tag{3.15}$$

where $\psi_x(s) = \hat{\varphi}^{\frac{1}{\gamma}}(s) \left(\frac{\Omega(s)}{\Omega(x)} \frac{R(x)}{R(s)} \beta(s-x) \right)^{\frac{1}{\gamma}}$ for all $s \in [x,T)$. Fifth, by plugging equations (3.14) and (3.15) into (3.12), the lagrangian multiplier satisfies:

$$\left(\frac{1}{\lambda(x)} \right)^{\frac{1}{\gamma}} = \frac{k(x) + \int_x^T \frac{R(s)}{R(x)} y(s) ds}{\int_x^T \frac{R(s)}{R(x)} \psi_x(s) \left(1 - \frac{1}{\gamma}\varphi(s,x)\frac{\theta^2(s)}{\eta^2} \right) ds}. \tag{3.16}$$

Sixth and last, we introduce (3.16) into (3.14). So, the rate of expenditure on consumption and the amount of money invested in risky assets at age x are equal to

$$c(x,x) = \psi_x(x) \frac{k(x) + \int_x^T \frac{R(s)}{R(x)} y(s) ds}{\int_x^T \frac{R(s)}{R(x)} \psi_x(s) \left(1 - \frac{1}{\gamma} \varphi(s,x) \frac{\theta^2(s)}{\eta^2}\right) ds}, \tag{3.17}$$

and

$$e(x,x) = \frac{1}{\gamma} \left(\frac{\alpha(x) - r(x)}{\sigma_\alpha^2(x)} \right) \frac{\varphi(x,x) c(x,x)}{\eta^2}. \tag{3.18}$$

Nonetheless, we still need to prove that (3.17) and (3.18) are maximums as well as (3.26) converges to a mean-variance utility function. Thus, $\Im$ satisfies the set of sufficient conditions for a regular interior maximum,

$$\begin{vmatrix} \Im_{\mathbf{cc}} & \Im_{\mathbf{ce}} \\ \Im_{\mathbf{ec}} & \Im_{\mathbf{ee}} \end{vmatrix} > 0,$$

where if $e(s,x) > 0$ (resp. < 0) then $\Im_{\mathbf{ce}} = \Im_{\mathbf{ec}} > 0$ (resp. < 0). And finally, following Tsiang (1972), we apply the following two constraints in order that a CRRA utility function converges to our mean-variance utility function:

1. $\frac{\sigma_c(s,x)}{c(s,x)} < \varepsilon, \forall s \in [x,T)$, where ε is an infinitesimal.

2. $1 - 2\frac{1+\gamma}{\gamma} \left(\frac{\theta(s)}{\eta}\right)^2 \geq 0, \forall s \in [x,T)$.

■

3.7 Appendix 2

It can be argued that the investment function presented here is more restrictive than the investment function from Merton (1971). In this Appendix, we will briefly demonstrate that the investment function introduced by Merton is also subject to similar constraints. Thus, it is first necessary to solve Merton's model with lifetime uncertainty.

3.7.1 *The Merton's Model*

In order to solve the optimal consumption and portfolio problem we are going to follow five steps. First, we will introduce the functional with life uncertainty. Second, we will present the budget constraint that our representative individual faces. Third, by using the Hamilton-Jacobi-Bellman method we will derive the optimization problem. In the fourth step, we will calculate the first optimal conditions and we will assume both a utility function and a functional which satisfy the necessary convergency conditions. Finally, in the fifth step, we will complete the proof by deriving the function $a(t)$, which will be introduced subsequently.

The Expected Utility Equation

The expected utility equation, or functional J, with lifetime uncertainty is:[16]

$$J(W(t),t) = \max_{\{c,w\}} \mathrm{E}_t \left(\int_t^T \frac{\dot{F}(s)}{F(t)} \left(\int_t^s U(c(\tau),\tau)d\tau + B(W(s),s) \right) ds \right).$$

Applying the Hamilton-Jacobi-Bellman approximation to our functional,

$$J(W(t),t) = \max_{\{c,w\}} \mathrm{E}_t \left(\int_t^T (\frac{F(s)}{F(t)} U(c(s),s) + \frac{\dot{F}(s)}{F(t)} B(W(s),s))ds \right),$$

$$J(W(t),t) = \max_{\{c,w\}} \mathrm{E}_t \left(\int_t^T \frac{F(s)}{F(t)} (U(c(s),s) + (\mu(s)+\delta)B(W(s),s))ds \right),$$

$$J(W(t),t) = \max\nolimits_{\{c,w\}} \mathrm{E}_t(U(c(t),t)dt + (\mu(t)+\delta)B(W(t),t)dt$$
$$+(1-(\mu(t)+\delta)dt)J(W(t)+dW(t),t+dt)).$$

[16] $F(t) = \Omega(t)\beta(t)$.

Thus, the functional $J(W(t)+dW(t),t+dt)$ can be approximated as follows[17]

$$J(W(t)+dW(t),t+dt) \approx J + J_t dt + J_W dW + \frac{1}{2} J_{WW}(dW)^2 + O(dt). \quad (3.19)$$

The Budget Constraint

The individual faces the following budget constraint:

$$dW(t) = (y(t)-c(t))dt + w(t)W(t)((\alpha(t)-r(t))dt + \sigma(t)dq(t)) + r(t)W(t)dt, \quad (3.20)$$

where its mean and variance operators are:

$$\mathrm{E}_t(dW(t)) = (y(t)-c(t))dt + (w(t)(\alpha(t)-r(t)) + r(t))W(t)dt, \quad (3.21)$$

$$\mathrm{E}_t(dW(t))^2 = \sigma^2(t)w^2(t)W^2(t)dt. \quad (3.22)$$

The Optimization Problem

After using the Hamilton-Jacobi-Bellman method we have the following equation to solve:

$$0 = \max_{\{c,w\}}(U(c(t),t)dt + (\mu(t)+\delta)(B(W(t),t) - J(W(t),t))dt + J_t dt + J_W dW + \frac{1}{2}J_{WW}(dW)^2), \quad (3.23)$$

or equivalently

$$\begin{aligned} 0 = \max\nolimits_{\{c,w\}}(U(c(t),t) + (\mu(t)+\delta)(B(W(t),t) - J) + J_t \\ + J_W((y(t)-c(t)) + (w(t)(\alpha(t)-r(t)) + r(t))W(t)) + \tfrac{1}{2}J_{WW}(\sigma^2(t)w^2(t)W^2(t)). \end{aligned}$$

Now Let us assume that our individual is a selfish consumer. Thus, the individual does not leave a bequest, $B(W(t),t) = 0, \forall t \in (0,T)$. As a consequence, the equation is as follows

$$\begin{aligned} 0 = \max\nolimits_{\{c,w\}}(U(c(t),t) - (\mu(t)+\delta)J + J_t + J_W((y(t)-c(t)) \\ +(w(t)(\alpha(t)-r(t)) + r(t))W(t)) \\ +\tfrac{1}{2}J_{WW}(\sigma^2(t)w^2(t)W^2(t)). \end{aligned}$$

[17] Hereinafter, for the sake of simplicity, we denote $J(W(t),t)$ as J.

First Optimal Conditions

The following equations give us the first optimal conditions:

$$U_c(c(t),t) - J_W = 0, \tag{3.24}$$

$$J_W(\alpha(t) - r(t)) + w(t)J_{WW}\sigma^2(t)W(t) = 0. \tag{3.25}$$

Now, assuming that the utility function is

$$U(c(t),t) = \frac{c(t)^{1-\gamma}}{1-\gamma} \tag{3.26}$$

and the functional is

$$J(W(t),t) = a(t)\frac{(W(t)+b(t))^{1-\gamma}}{1-\gamma}, \tag{3.27}$$

where $b(t) = \int_t^T y(s)\mathrm{e}^{-\int_t^s r(\tau)d\tau}ds$. Then, by plugging (3.26) and (3.27) into both (3.24) and (3.25), the optimal consumption and investment in risky asset are respectively:

$$c^*(t) = (W(t)+b(t))a(t)^{-\frac{1}{\gamma}}, \tag{3.28}$$

$$w^*(t)W(t) = \frac{1}{\gamma}\left(\frac{\alpha(t)-r(t)}{\sigma^2(t)}\right)(W(t)+b(t)). \tag{3.29}$$

Function $a(t)$

Thus far, we have obtained the solution to the equation, however in order to complete the demonstration, we need to derive the function $a(t)$. Thus,

$$0 = Ja(t)^{-\frac{1}{\gamma}} - (\mu(t)+\delta)J + J_t + J_W(y(t) - c(t) + (w(t)(\alpha(t)-r(t)) + r(t))W(t)) \\ +\tfrac{1}{2}J_{WW}\sigma^2(t)w^2(t)W^2(t) \Rightarrow$$

$$\Rightarrow 0 = Ja(t)^{-\frac{1}{\gamma}} - (\mu(t)+\delta)J + J(\tfrac{\dot{a}(t)}{a(t)} + (1-\gamma)\tfrac{\dot{W}(t)+\dot{b}(t)}{W(t)+b(t)}) + \tfrac{1-\gamma}{W(t)+b(t)}J(y(t) - c(t) + \\ (w(t)(\alpha(t)-r(t)) + r(t))W(t)) - \tfrac{1}{2}\tfrac{\gamma(1-\gamma)}{(W(t)+b(t))^2}J\sigma^2(t)w^2(t)W^2(t).$$

Dividing this equation by the common factor J, it gives that

$$0 = a(t)^{-\frac{1}{\gamma}} - \mu(t) - \delta + \tfrac{\dot{a}(t)}{a(t)} + (1-\gamma)\tfrac{\dot{W}(t)+\dot{b}(t)}{W(t)+b(t)} + \tfrac{1-\gamma}{W(t)+b(t)}(y(t) - c(t) + (w(t)(\alpha(t) - \\ r(t)) + r(t))W(t)) - \tfrac{1}{2}\tfrac{\gamma(1-\gamma)}{(W(t)+b(t))^2}\sigma^2(t)w^2(t)W^2(t).$$

Subsequently, by using equations (3.28) and (3.29), we get that

$$0 = a(t)^{-\frac{1}{\gamma}} - \mu(t) - \delta + \tfrac{\dot{a}(t)}{a(t)} + (1-\gamma)\tfrac{\dot{W}(t)+\dot{b}(t)}{W(t)+b(t)} - (1-\gamma)a(t)^{-\frac{1}{\gamma}} + \tfrac{1-\gamma}{W(t)+b(t)}(y(t) + \tfrac{1}{\gamma}\theta^2(t)(W(t)+b(t)) + r(t)W(t)) - \tfrac{1}{2}\tfrac{(1-\gamma)}{\gamma}\theta^2(t),$$

where $\theta(t) = \frac{\alpha(t)-r(t)}{\sigma(t)}$. Now, we proceed to simplify

$$0 = \gamma a(t)^{-\frac{1}{\gamma}} - \mu(t) - \delta + \tfrac{\dot{a}(t)}{a(t)} + \tfrac{(1-\gamma)}{W(t)+b(t)}(\dot{W}(t) + \dot{b}(t)) + \tfrac{1-\gamma}{W(t)+b(t)}(y(t) + r(t)W(t)) + \tfrac{1}{2}\tfrac{(1-\gamma)}{\gamma}\theta^2(t).$$

From (3.21), if $\dot{W}(t)$ is equal to $y(t) - c(t) + w(t)W(t)\theta(t)\sigma(t) + r(t)W(t)$ and $\dot{b}(t) = -y(t) + r(t)b(t)$, the equation will give

$$0 = (2\gamma-1)a(t)^{-\frac{1}{\gamma}} - \mu(t) - \delta + \tfrac{\dot{a}(t)}{a(t)} + (1-\gamma)r(t) + \tfrac{1-\gamma}{W(t)+b(t)}(y(t)+r(t)W(t)) + \tfrac{3}{2}\tfrac{(1-\gamma)}{\gamma}\theta^2(t).$$

Now, let us assume that:

$$k(t) = -\mu(t) - \delta + (1-\gamma)r(t) + \frac{1-\gamma}{W(t)+b(t)}(y(t)+r(t)W(t)) + \frac{3}{2}\frac{(1-\gamma)}{\gamma}\theta^2(t). \quad (3.30)$$

As a consequence,

$$0 = (2\gamma-1)a(t)^{-\frac{1}{\gamma}} + \frac{\dot{a}(t)}{a(t)} + k(t).$$

Multiplying the latter equation by $a(t)$, then $\dot{a}(t)$ results

$$\dot{a}(t) = (1-2\gamma)a(t)^{1-\frac{1}{\gamma}} - k(t)a(t). \quad (3.31)$$

We solve $\dot{a}(t)$ applying Bernouilly techniques:

$$z(t) = a(t)^{\frac{1}{\gamma}} \Rightarrow$$
$$\dot{z}(t) = a(t)^{\frac{1}{\gamma}-1}\dot{a}(t) \Rightarrow$$
$$\dot{z}(t) = a(t)^{\frac{1}{\gamma}-1}((1-2\gamma)a(t)^{1-\frac{1}{\gamma}} - k(t)a(t)) \Rightarrow$$
$$\dot{z}(t) = (1-2\gamma) - k(t)z(t).$$

By calculating the homogenous case of $z(t)$,

$$\boxed{\dot{z}(t) = -k(t)z(t)}$$

then

$$\int_t^T \frac{dz(s)}{z(s)} = -\int_t^T k(s)ds \Rightarrow z(t) = z(T)\mathrm{e}^{\int_t^T k(s)ds}.$$

Now, we proceed to calculate a particular case of $z(t)$,

$$\boxed{z(t) = C(t)\mathrm{e}^{\int_t^T k(s)ds}}$$

then

$$\dot{z}(t) = \dot{C}(t)\mathrm{e}^{\int_t^T k(s)ds} - k(t)C(t)\mathrm{e}^{\int_t^T k(s)ds} = (1-2\gamma) - k(t)z(t).$$

Thus,

$$\begin{gathered}
\dot{C}(t)\mathrm{e}^{\int_t^T k(s)ds} = (1-2\gamma) \Rightarrow \\
\dot{C}(t) = (1-2\gamma)\mathrm{e}^{-\int_t^T k(s)ds} \Rightarrow \\
\int_t^T dC(t) = (1-2\gamma)\int_t^T \mathrm{e}^{-\int_s^T k(\tau)d\tau}ds \Rightarrow \\
C(t) = C(T) - (1-2\gamma)\int_t^T \mathrm{e}^{-\int_s^T k(\tau)d\tau}ds.
\end{gathered}$$

Therefore,

$$\begin{gathered}
z(t) = (C(T) - (1-2\gamma)\int_t^T \mathrm{e}^{-\int_s^T k(\tau)d\tau}ds)\mathrm{e}^{\int_t^T k(s)ds} \Rightarrow \\
z(t) = C(T)\mathrm{e}^{\int_t^T k(s)ds} - (1-2\gamma)\int_t^T \mathrm{e}^{\int_t^s k(\tau)d\tau}ds.
\end{gathered}$$

Finally, we have assumed a selfish investor, which implies that $z(T) = 1$. Then,

$$\boxed{z(t) = \mathrm{e}^{\int_t^T k(s)ds} - (1-2\gamma)\int_t^T \mathrm{e}^{\int_t^s k(\tau)d\tau}ds} \tag{3.32}$$

3.7.2 *Analysis of solutions*

The optimal consumption and investment in risky assets have the following mapping:

$$\boxed{c^*(t) = \frac{W(t)+b(t)}{\mathrm{e}^{\int_t^T k(s)ds} - (1-2\gamma)\int_t^T \mathrm{e}^{\int_t^s k(\tau)d\tau}ds}} \tag{3.33}$$

$$\boxed{w^*(t)W(t) = \frac{1}{\gamma}\left(\frac{\alpha(t)-r(t)}{\sigma^2(t)}\right)(W(t)+b(t))} \tag{3.34}$$

Now, we are going to calculate the marginal propensity to consume as a function of time. Firstly, in order to know how $z(t)$ evolves, we only need to obtain the first time and the last time points within {0,T}, since $z(t)$ is a monotonic function. Thus,

$$\lim_{t\to 0} c^*(t) = \frac{W(0)+b(0)}{\mathrm{e}^{\int_0^T k(s)ds} - (1-2\gamma)\int_0^T \mathrm{e}^{\int_0^s k(\tau)d\tau}ds}, \forall k(s), \tag{3.35}$$

and

$$\lim_{t \to T} c^*(t) = W(T) + b(T), \forall k(s). \tag{3.36}$$

For the sake of simplicity, we will consider the case when $k(s) = k, \forall s$. Therefore the first limit is

$$\lim_{t \to 0} c^*(t) = \frac{W(0) + b(0)}{(1 - \frac{1-2\gamma}{k})e^{kT} + \frac{1-2\gamma}{k}}. \tag{3.37}$$

By assuming that $T \to \infty$, k should be a negative scalar in order to have an economic meaning; otherwise, the marginal propensity to consume will be zero. Hence, the following constraints apply:

a) $\gamma > \frac{1}{2}$.

b) $k > 1 - 2\gamma$.

Using (3.30) with the latter constraint

$$\text{c)} \begin{cases} \text{if} \quad \frac{1}{2} < \gamma < 1 \quad \theta^2 > \frac{2\gamma}{3}\left(1 - r - \frac{y+rW}{W+b} - \frac{\gamma-\mu-\delta}{1-\gamma}\right) \\ \\ \text{if} \qquad \gamma > 1 \quad \theta^2 < \frac{2\gamma}{3}\left(1 - r - \frac{y+rW}{W+b} - \frac{\gamma-\mu-\delta}{1-\gamma}\right) \end{cases}$$

In addition to the latter constraints, it is necessary to pay attention to the saving path. To do so, we need to demonstrate that the hump saving pattern applies. Hence, from (3.21) we know that

$$\frac{\dot{W}(t) + \dot{b}(t)}{W(t) + b(t)} = -\frac{k}{1 - 2\gamma} + \frac{1}{\gamma}\theta^2 + r.$$

For the sake of simplicity, let us assume that $y(s) = 0, \forall s$ and in average θ is equal to 0, thus

$$\frac{\dot{W}(t)}{W(t)} = r - \frac{k}{1 - 2\gamma}.$$

Therefore, if we are looking for a hump saving pattern, it is needed that $\frac{\dot{W}(t)}{W(t)} > 0$ for $t \in [0, \bar{T})$, where $\bar{T}$ corresponds at least to half of the total life span.

d) $\mu + \delta < r$.

In sum, following Merton (1971) we need to satisfy these four conditions in order to give an economic meaning, rather than the two conditions introduced in this thesis.

3.8 Simulations

In Chapter 2 we analyzed the desire to purchase annuities at the age of retirement, $J = 65$. We considered two initial wealths (150.000 and 300.000) and a flat pension benefit, paid by an unfunded social security, which depends on the demographic structure as well as on the factor prices under stationary conditions.

Here we extend the analysis to the life cycle, instead of simply the retirement period. We thus determine how both consumption and wealth evolve along the life cycle. Nonetheless, note that current wealth profiles can be split into private and public wealths. Thus, in order to compare the current results with those obtained in Chapter 2, we need to take into account that these wealth paths include both public and private data. On the other hand, because we have already proven that the survival probability does not completely explain the demand for private annuities at a given age, we have only simulated two Spanish cohorts and have also focused our simulations on the male case. For the sake of reality, we have estimated a non-flat gross earning profile following a Mincerian equation, since we are limited to the average salary of the economy from Chapter 4. Finally, it is worth noting that in the current chapter the social security system is funded.

These simulations proceed as follows: first, we compare the allocation process for different survival probabilities {1960 and 2000}. Second, we change with respect to previous simulations the age of retirement from 65, from previous simulations, to 70. Third, we study the allocation process under different factor prices. We will choose each alternative according to Table 3.6 below.

We have found that the greater the life expectancy of the individual, the greater the need to accumulate assets for retirement and thus, the greater the likelihood that she will purchase annuities. In the same way, the greater the life expectancy is, the lower the effect of a funded social security on the demand for annuities becomes. Thus, in this particular case, a payroll tax of 3 percent does not affect on the demand for annuities when the individual belongs to the cohort were born in 2000. Thus, even though we claimed in Chapter 2 that the survival probability does not affect the demand for annuities at a certain age, it is not absolutely true in a life-cycle context. We can expect therefore that an economy populated by individuals with a greater life expectancy will not only save more, but will also the population purchase more annuities.

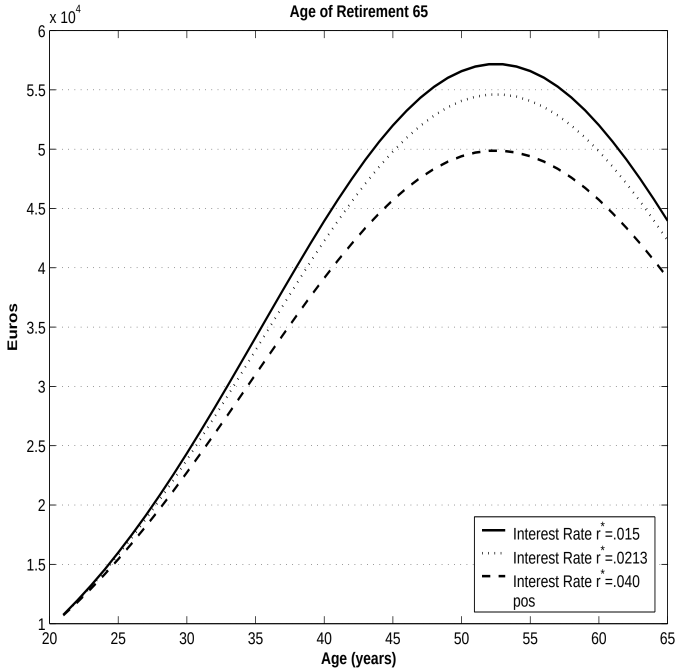
We have also found that the delaying of the age of retirement has a negative impact on the demand for private annuities. The motive for this is twofold. First of all, the number of years of contribution to the funded social security raises and as a consequence the pension benefit increases. Seconds, the life expectancy at retirement decreases, and so the pension benefit increases as well. Hence, the individual is less willing to insure her wealth as higher the age of retirement. Thus, the impact of this policy cannot be well studied under a partial equilibrium model, since it gives ambiguous results due to the negative impact on the demand for annuities, e.g. Tables 3.7 and 3.9.

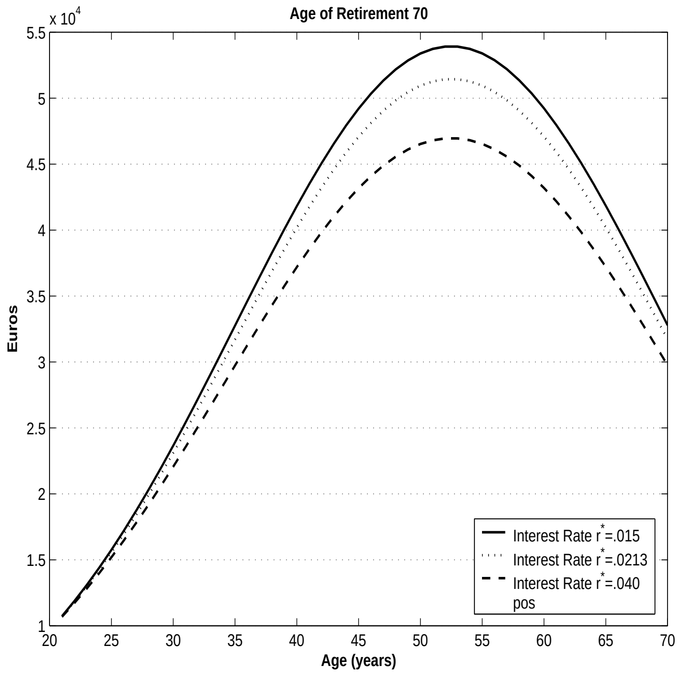
The risk aversion, similar to the survival probability, is able to modify the decision of purchasing an asset contingent on her mortality risk, but only throughout a life-cycle framework. At the beginning of the life cycle, the consumption of our individual is greater under the condition that $\gamma = 2$ than $\gamma = .75$. However, the former yields a greater saving afterwards. Consequently, it is likely that as greater the risk aversion becomes, the individual is more willing to purchase annuities.

Table 3.6: Stationary Wage Profiles According to both the Age of Retirement and the Cohort

	Retirement Age	65			
Risk Aversion Coefficient γ	Cohort	1960	β_1	2000	β_1
.75	.0150	19.180	0.10785	19.180	0.10779
	.0213	18.474	0.10494	18.474	0.10490
2	.0150	19.180	0.10785	19.180	0.10779
	.0400	17.179	0.09927	17.179	0.09928
	Retirement Age	70			
Risk Aversion Coefficient γ	Cohort	1960	β_1	2000	β_1
.75	.0150	19.180	0.10414	19.180	0.10334
	.0213	18.474	0.10120	18.474	0.10042
2	.0150	19.180	0.10414	19.180	0.10334
	.0400	17.179	0.09549	17.179	0.09474

Figure 3.7: Gross Salary Profiles





We have used the following Mincerian equation with the β_1 obtained from Table 3.6:

$$w(s) = 10.000 \cdot \mathrm{e}^{\beta_1(s-20)-\frac{\beta_1}{2(52-20)}(s-20)^2}, \forall s \in [20, 70].$$

Table 3.7: Wealth by Age: Cohort 1960 (Men). Case $J = 65$, $\gamma = .75$, and $w = 19.180$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-129.0910	-38.008	10.809	14.412	-152.257	5.405	10.809	14.412
40	-112.591	-34.112	35.391	47.188	-158.617	17.695	35.391	47.188
50	66.842	130.921	77.944	103.926	941	49.338	77.944	103.926
60	324.912	373.805	142.902	190.537	238.764	141.999	142.902	190.537
70	377.312	411.641	154.195	205.593	254.623	77.060	154.195	205.593
80	242.730	264.678	94.286	125.715	99.566	47.100	94.286	125.715
90	143.126	25.323	52.323	52.506	70.009	19.735	26.203	52.506
100	78.941	12.413	26.985	35.980	974	13.434	26.985	35.980

Figure 3.8: Consumption and A.E.W. by Age: Cohort 1960 (Men). $J = 65$, $\gamma = .75$, and $w = 19.180$

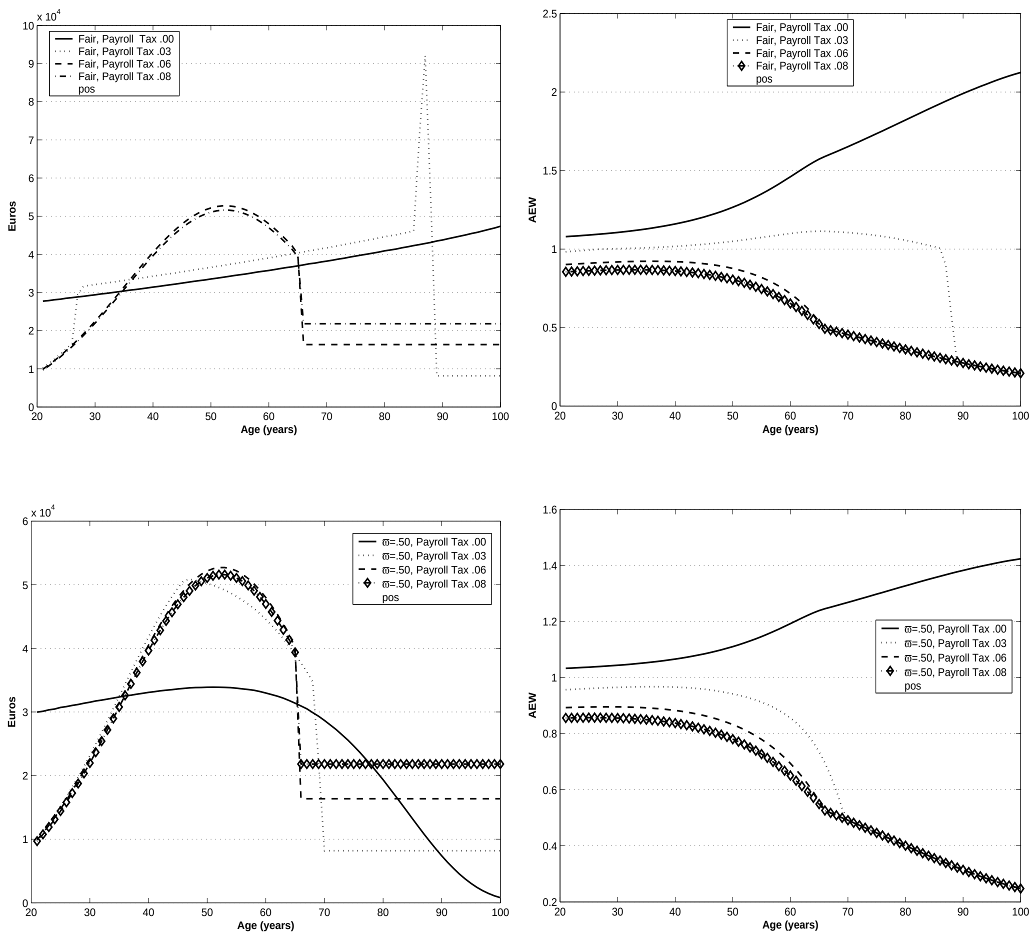


Table 3.8: Wealth by Age: Cohort 2000 (Men). Case $J = 65$, $\gamma = .75$, and $w = 19.180$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-111.354	-111.342	10.762	14.349	-123.621	-39.987	10.762	14.349
40	-70.976	-70.955	51.759	46.514	-96.524	-28.093	34.886	46.514
50	137.066	137.092	240.381	112.560	97.789	147.691	117.733	112.560
60	416.240	416.261	497.491	219.580	359.433	388.715	254.700	219.580
70	455.964	455.931	514.350	179.025	366.066	297.507	140.906	179.025
80	294.444	294.355	103.507	110.026	173.591	41.355	82.822	110.026
90	168.902	168.723	44.414	59.553	46.972	22.443	45.016	59.553
100	87.186	86.827	21.037	28.438	3.750	10.792	21.736	28.438

Figure 3.9: Consumption and A.E.W. by Age: Cohort 2000 (Men). $J = 65$, $\gamma = .75$, and $w = 19.180$

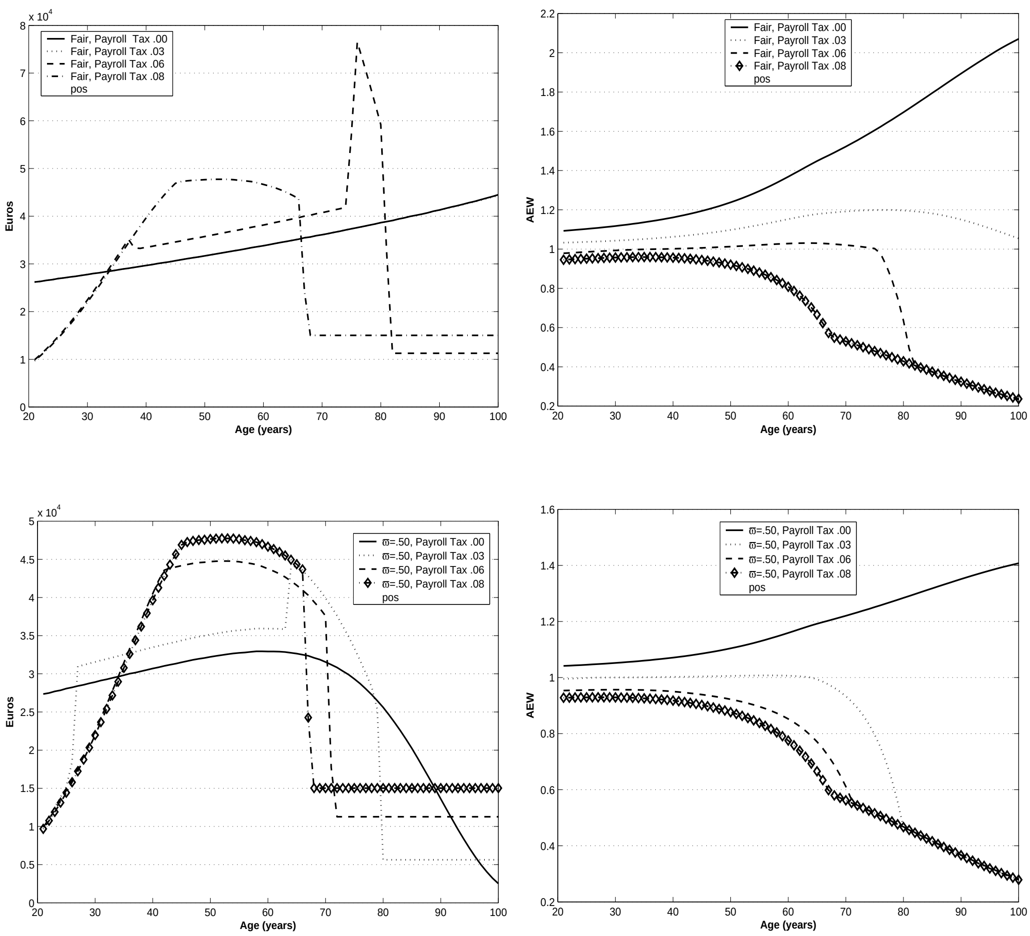


Table 3.9: Wealth by Age: Cohort 1960 (Men). Case $J = 70$, $\gamma = .75$, and $w = 19.180$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-133.551	5.307	10.614	14.152	-160.026	5.307	10.614	14.152
40	-134.683	17.115	34.229	45.639	-187.908	17.115	34.229	45.639
50	7.464	37.335	74.670	99.560	-69.260	37.335	74.670	99.560
60	207.794	68.180	136.359	181.812	111.639	68.180	136.359	181.812
70	378.669	124.763	249.527	332.703	258.032	124.763	249.527	332.703
80	243.603	76.290	152.580	203.439	100.899	76.290	152.580	203.439
90	143.641	42.485	84.969	113.292	20.000	42.485	84.969	113.292
100	79.225	21.835	43.669	58.226	987	21.835	43.669	58.226

Figure 3.10: Consumption and A.E.W. by Age: Cohort 1960 (Men). $J = 70$, $\gamma = .75$, and $w = 19.180$

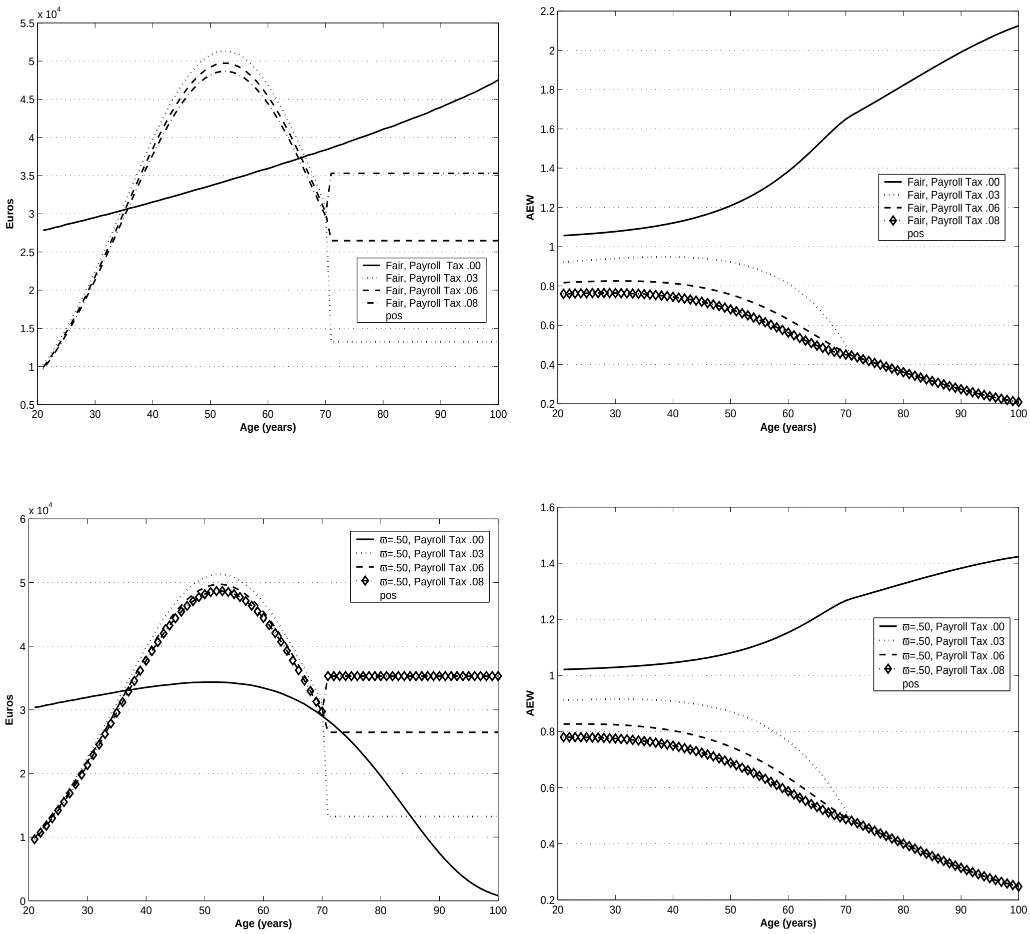


Table 3.10: WEALTH BY AGE: COHORT 2000 (MEN). CASE $J = 70$, $\gamma = .75$, AND $w = 19.180$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-116.271	-110.772	10.526	14.034	-130.225	5.263	10.526	14.034
40	-96.344	-91.469	33.495	44.661	-125.574	16.748	33.495	44.661
50	69.438	73.546	71.185	94.913	24.621	70.508	71.185	94.913
60	289.483	292.709	122.672	163.563	227.740	165.790	122.672	163.563
70	457.456	459.745	197.205	262.940	369.274	203.014	197.205	262.940
80	295.408	296.786	121.453	161.938	175.112	60.424	121.453	161.938
90	169.454	32.118	66.074	88.099	47.384	32.686	66.074	88.099
100	87.471	14.923	31.981	42.641	3.783	15.582	31.981	42.641

Figure 3.11: CONSUMPTION AND A.E.W. BY AGE: COHORT 2000 (MEN). $J = 70$, $\gamma = .75$, AND $w = 19.180$

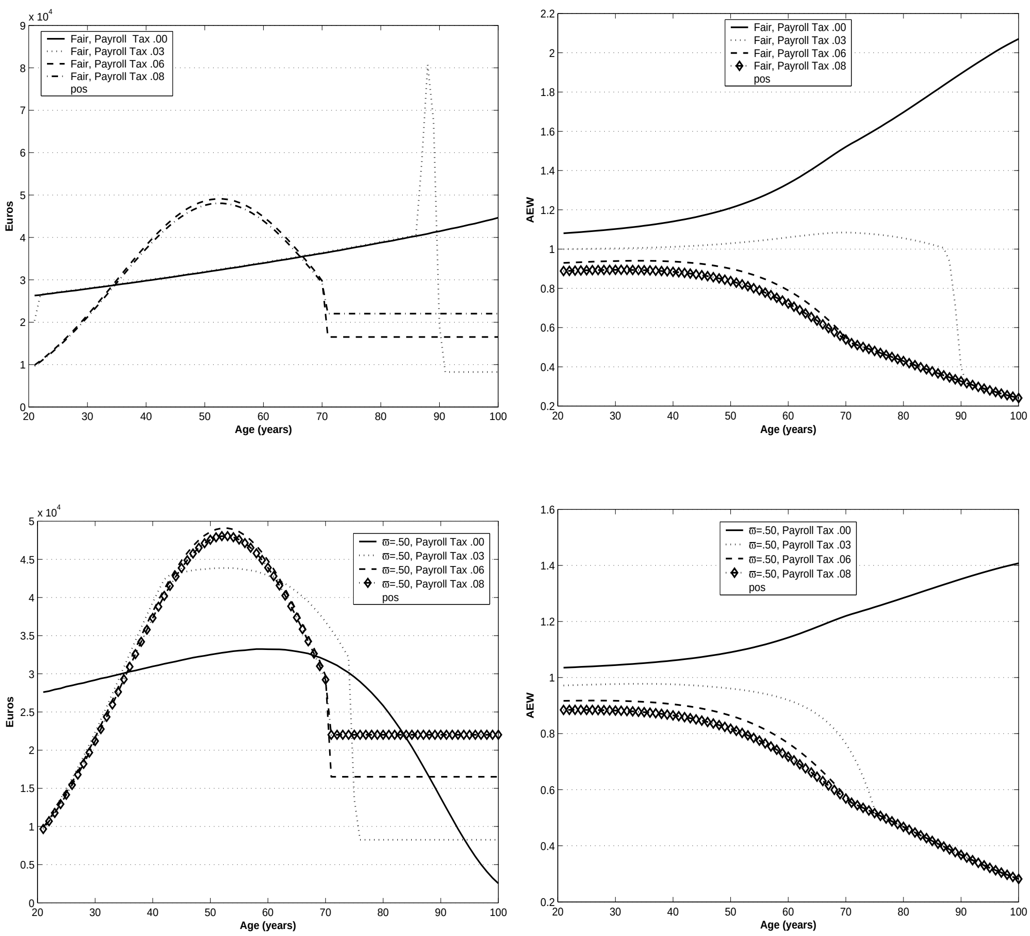


Table 3.11: Wealth by Age: Cohort 1960 (Men). Case $J = 65$, $\gamma = .75$, and $w = 18.474$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-81.078	-81.067	10.950	14.600	-100.482	5.475	10.950	14.600
40	-37.821	-37.803	36.294	48.392	-79.752	18.147	36.294	48.392
50	148.119	148.138	81.478	108.637	81.695	72.234	81.478	108.637
60	402.531	402.540	154.072	205.429	305.407	174.749	154.072	205.429
70	457.223	457.142	171.112	228.149	307.756	85.513	171.112	228.149
80	317.574	317.348	106.146	141.528	129.972	53.020	106.146	141.528
90	202.425	201.976	59.766	79.688	27.862	29.818	59.766	79.688
100	120.828	13.541	30.947	41.263	1.489	15.393	30.947	41.263

Figure 3.12: Consumption and A.E.W. by Age: Cohort 1960 (Men). $J = 65$, $\gamma = .75$, and $w = 18.474$

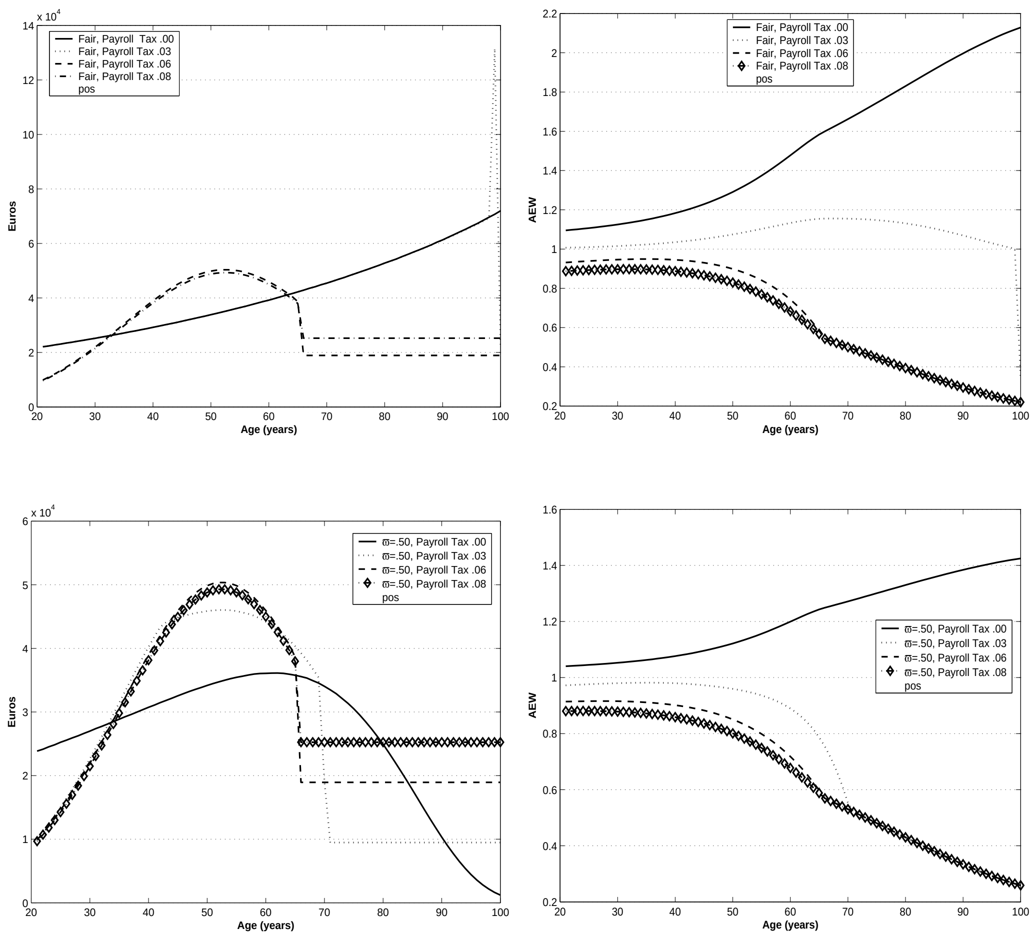


Table 3.12: WEALTH BY AGE: COHORT 2000 (MEN). CASE $J = 65$, $\gamma = .75$, AND $w = 18.474$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-63.821	-63.809	-63.796	14.533	-74.204	-76.914	10.900	14.533
40	4.482	4.505	4.527	47.673	-18.991	-25.432	39.173	47.673
50	222.051	222.079	222.107	143.808	182.327	171.071	155.972	143.808
60	502.327	502.349	502.372	267.562	438.610	422.183	305.224	267.562
70	548.972	548.934	548.896	198.307	440.259	407.717	191.168	198.307
80	382.394	382.282	382.170	124.037	225.256	71.093	92.772	124.037
90	236.918	236.692	50.236	68.147	65.858	25.395	50.795	68.147
100	132.275	131.826	23.639	32.958	5.690	12.162	24.329	32.958

Figure 3.13: CONSUMPTION AND A.E.W. BY AGE: COHORT 2000 (MEN). $J = 65$, $\gamma = .75$, AND $w = 18.474$

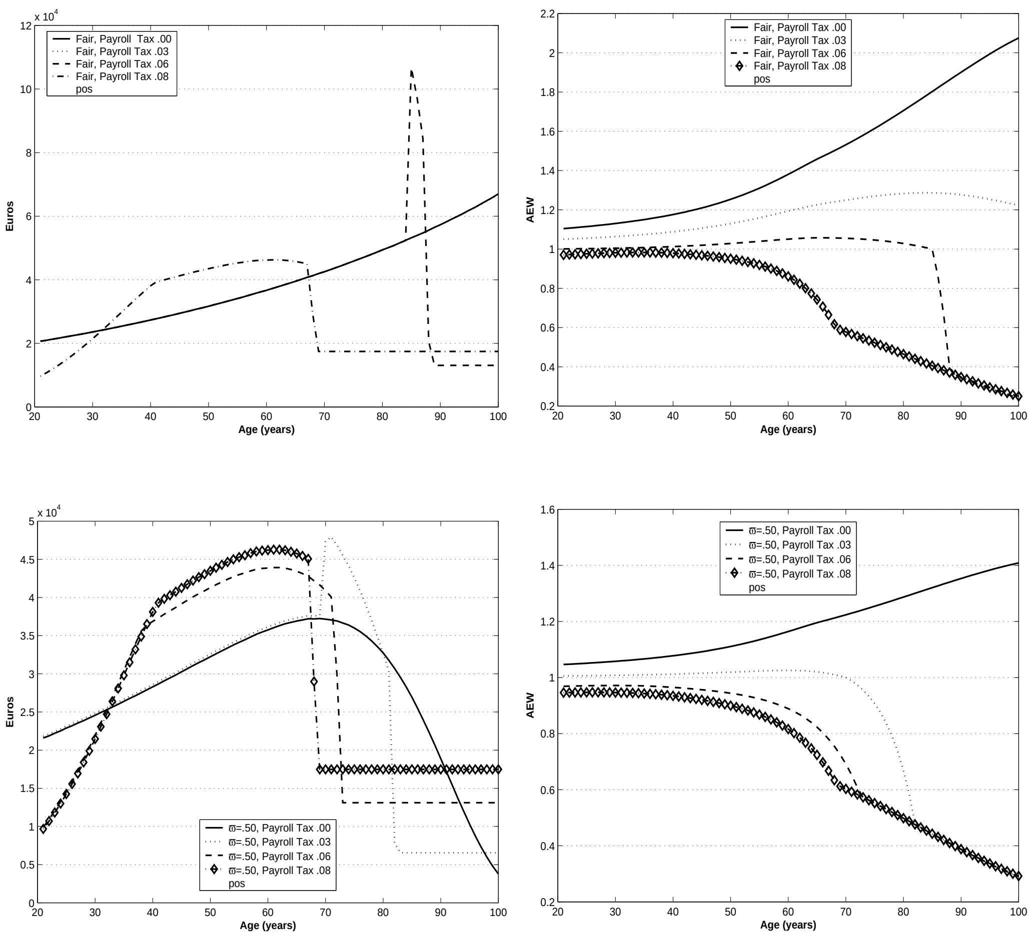


Table 3.13: Wealth by Age: Cohort 1960 (Men). Case $J = 70$, $\gamma = .75$, and $w = 18.474$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-83.633	5.375	10.750	14.334	-105.371	5.375	10.750	14.334
40	-55.849	17.549	35.099	46.798	-103.251	17.549	35.099	46.798
50	94.604	39.027	78.054	104.071	19.391	39.027	78.054	104.071
60	291.288	73.509	147.019	196.025	185.774	73.509	147.019	196.025
70	455.760	140.627	281.254	375.006	309.397	140.627	281.254	375.006
80	316.557	87.236	174.471	232.629	130.665	87.236	174.471	232.629
90	201.777	49.118	98.237	130.982	28.010	49.118	98.237	130.982
100	120.441	25.434	50.867	67.823	1.497	25.434	50.867	67.823

Figure 3.14: Consumption and A.E.W. by Age: Cohort 1960 (Men). $J = 70$, $\gamma = .75$, and $w = 18.474$

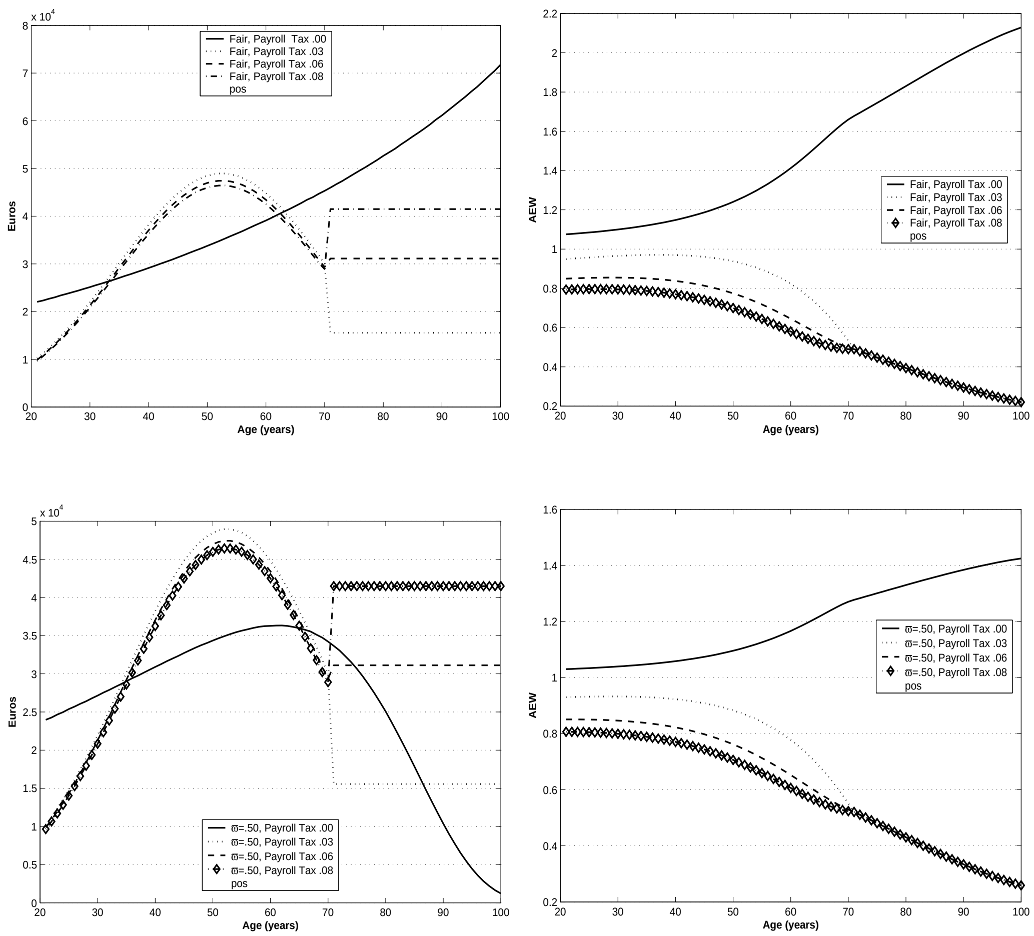


Table 3.14: Wealth by Age: Cohort 2000 (Men). Case $J = 70$, $\gamma = .75$, and $w = 18.474$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-66.857	-66.846	10.662	14.216	-78.417	5.331	10.662	14.216
40	-16.759	-16.741	34.345	45.794	-42.993	20.473	34.345	45.794
50	160.405	160.425	82.461	99.153	116.261	108.501	82.461	99.153
60	381.155	381.166	133.142	175.996	313.473	215.781	133.142	175.996
70	546.842	546.816	221.263	295.140	440.664	244.926	221.263	295.140
80	380.911	380.765	138.541	184.872	225.463	69.190	138.541	184.872
90	235.999	235.677	76.317	101.942	65.918	38.059	76.317	101.942
100	131.762	131.102	37.182	49.805	5.696	18.468	37.182	49.805

Figure 3.15: Consumption and A.E.W. by Age: Cohort 2000 (Men). $J = 70$, $\gamma = .75$, and $w = 18.474$

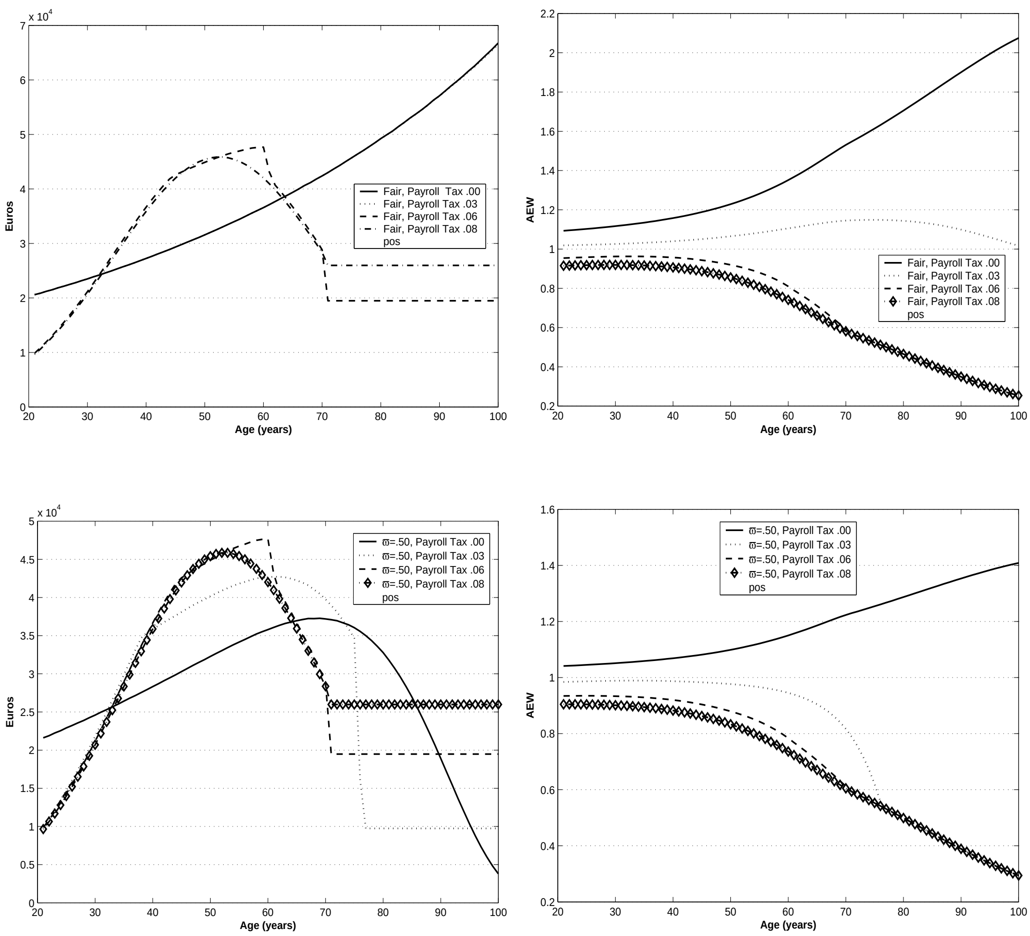


Table 3.15: Wealth by Age: Cohort 1960 (Men). Case $J = 65$, $\gamma = 2$, and $w = 19.180$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-154.624	-154.615	10.809	14.412	-153.265	-136.385	10.809	14.412
40	-156.130	-156.114	35.391	47.188	-149.496	-142.728	35.391	47.188
50	13.914	13.931	96.399	103.926	30.762	26.262	96.399	103.926
60	271.202	271.210	218.717	190.537	295.443	280.317	218.717	190.537
70	330.012	329.941	154.018	205.593	333.657	285.673	154.018	205.593
80	206.136	205.956	94.081	125.715	179.780	46.876	94.081	125.715
90	117.994	117.641	52.268	70.009	70.742	25.943	52.268	70.009
100	63.728	63.045	26.709	35.980	15.678	13.133	26.709	35.980

Figure 3.16: Consumption and A.E.W. by Age: Cohort 1960 (Men). $J = 65$, $\gamma = 2$, and $w = 19.180$

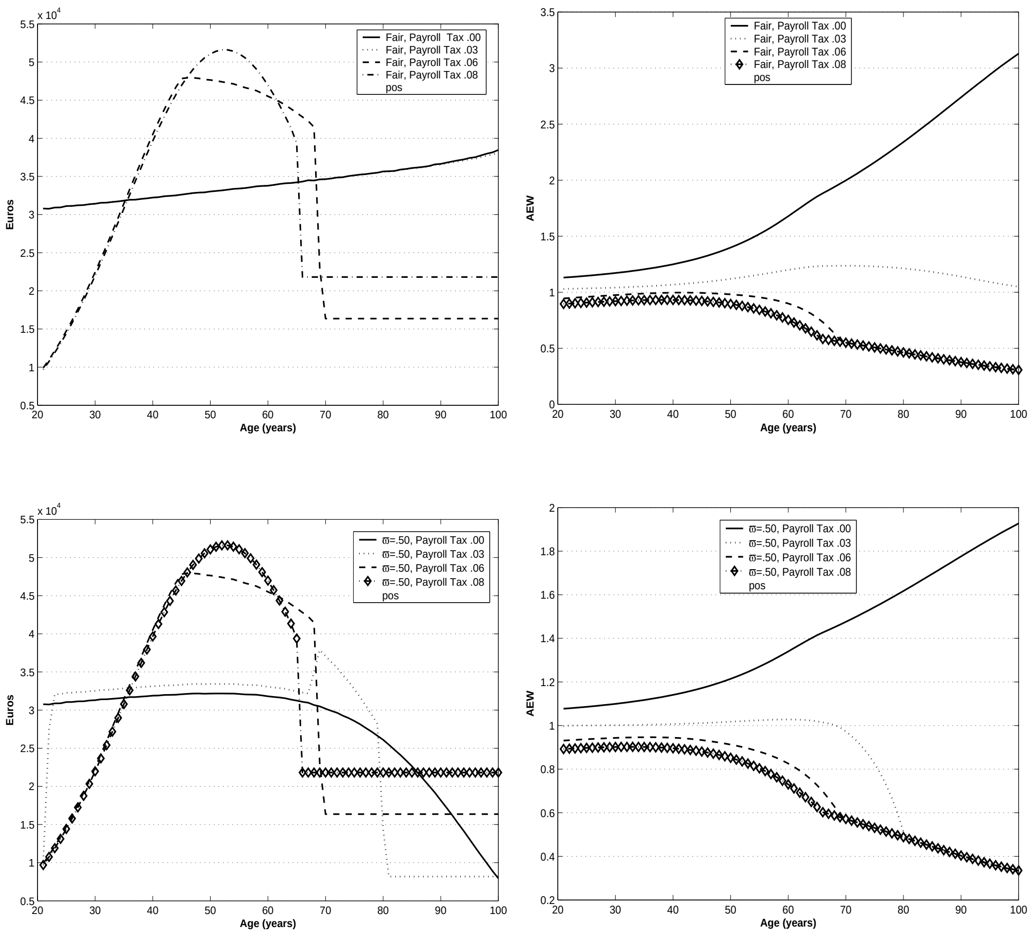


Table 3.16: Wealth by Age: Cohort 2000 (Men). Case $J = 65$, $\gamma = 2$, and $w = 19.180$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-138.670	-138.658	-138.647	-3.284	-133.438	-136.929	10.762	14.349
40	-118.607	-118.586	-118.566	-1.747	-105.975	-113.572	41.906	46.514
50	77.683	77.709	77.734	173.582	100.137	87.920	210.925	165.304
60	354.462	354.482	354.502	427.943	384.144	367.604	455.335	349.716
70	400.545	400.511	400.477	451.900	418.139	400.354	377.939	273.739
80	251.447	251.358	251.268	142.680	244.034	227.960	95.016	110.478
90	140.054	139.877	52.626	59.871	107.081	36.271	44.852	60.079
100	70.544	70.190	21.379	28.806	27.577	10.763	21.545	29.047

Figure 3.17: Consumption and A.E.W. by Age: Cohort 2000 (Men). $J = 65$, $\gamma = 2$, and $w = 19.180$

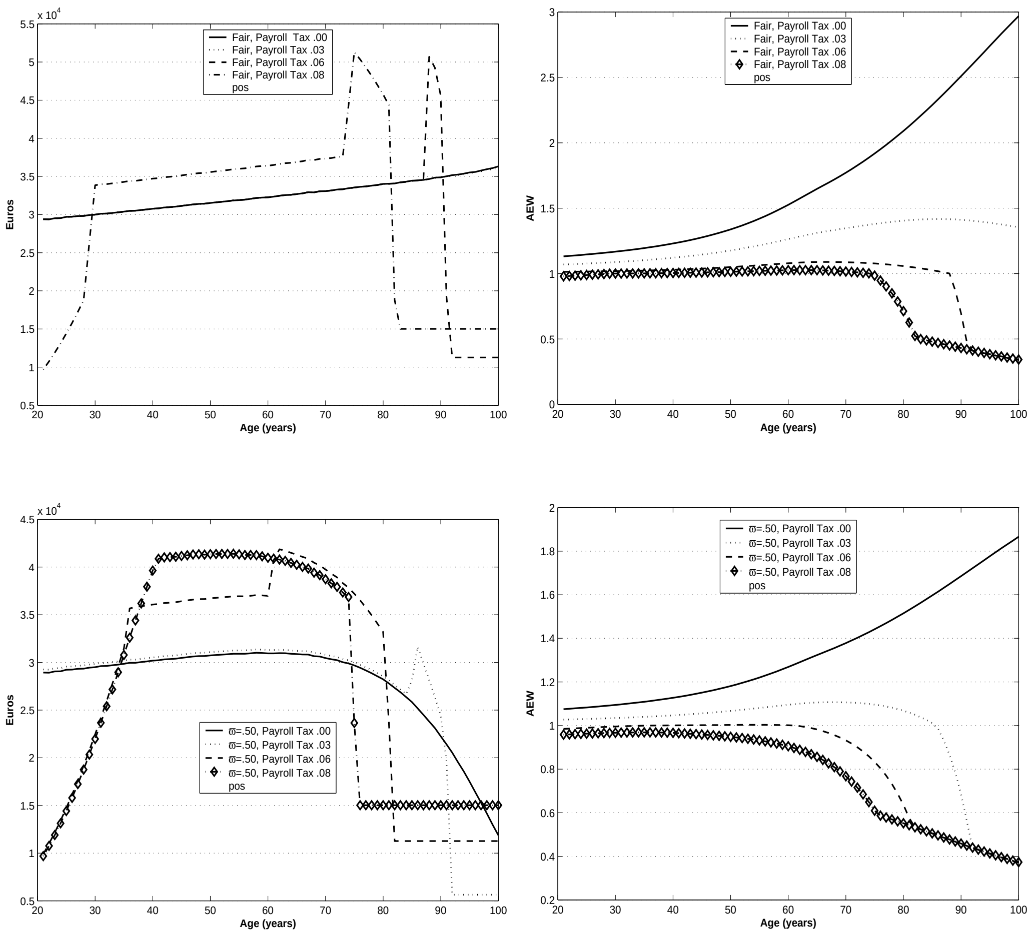


Table 3.17: Wealth by Age: Cohort 1960 (Men). Case $J = 70$, $\gamma = 2$, and $w = 19.180$

Age	Fair				$\varpi = .50$			
	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-159.186	-904	10.614	14.152	-161.048	5.307	10.614	14.152
40	-178.395	-45.527	34.229	45.639	-178.662	17.115	34.229	45.639
50	-45.675	59.980	74.670	99.560	-39.031	54.884	74.670	99.560
60	153.870	232.313	136.359	181.812	169.097	140.448	136.359	181.812
70	331.181	384.785	249.527	332.703	338.148	180.764	249.527	332.703
80	206.867	240.128	152.580	203.439	182.200	75.878	152.580	203.439
90	118.412	42.197	84.969	113.292	71.694	42.006	84.969	113.292
100	63.953	21.501	43.669	58.226	15.889	21.280	43.669	58.226

Figure 3.18: Consumption and A.E.W. by Age: Cohort 1960 (Men). $J = 70$, $\gamma = 2$, and $w = 19.180$

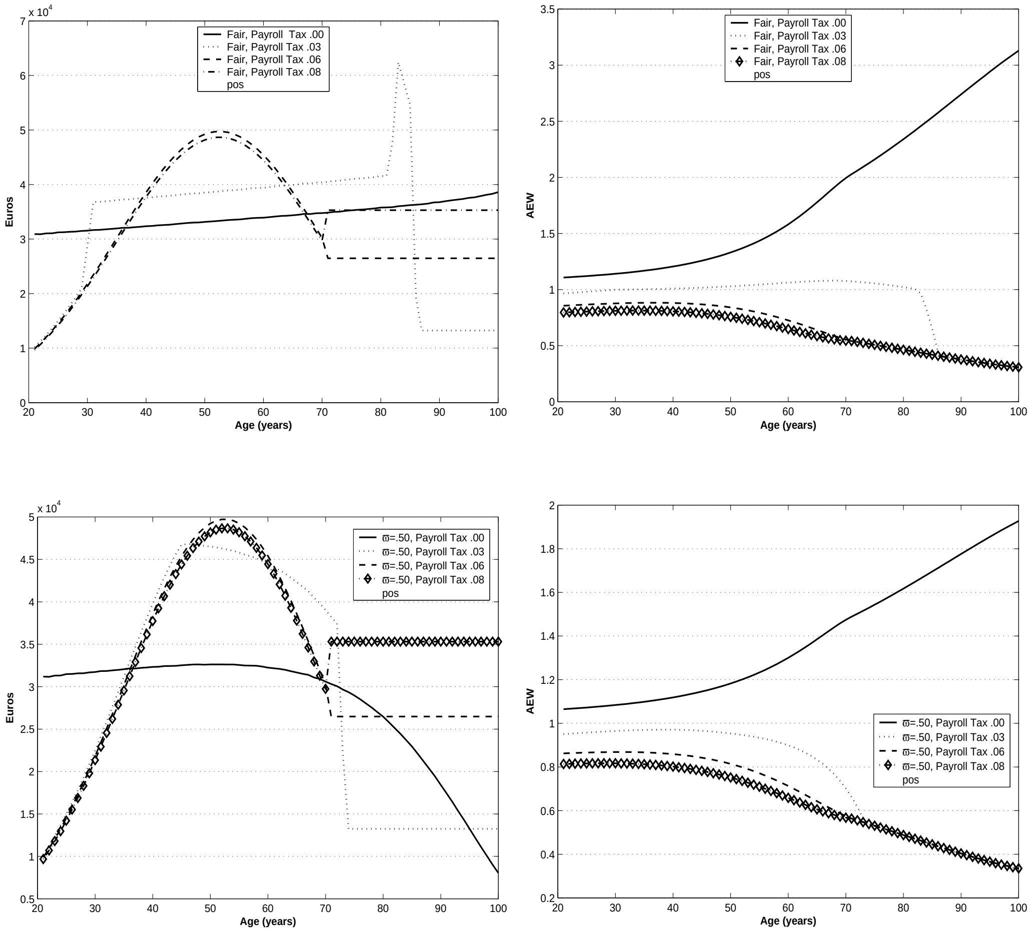


Table 3.18: Wealth by Age: Cohort 2000 (Men). Case $J = 70$, $\gamma = 2$, and $w = 19.180$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-143.681	-143.672	10.526	14.034	-140.129	-144.792	10.526	14.034
40	-144.140	-144.125	33.495	44.661	-135.109	-145.279	33.495	44.661
50	9.850	9.867	101.034	97.882	26.990	10.496	101.034	97.882
60	227.491	227.499	215.282	175.435	252.672	229.630	215.282	175.435
70	401.845	401.818	263.231	262.822	421.811	395.146	263.231	262.822
80	252.263	252.146	121.153	161.801	246.177	148.090	121.153	161.801
90	140.508	140.258	65.726	87.940	108.021	32.904	65.726	87.940
100	70.773	70.259	31.577	42.457	27.819	15.836	31.577	42.457

Figure 3.19: Consumption and A.E.W. by Age: Cohort 2000 (Men). $J = 70$, $\gamma = 2$, and $w = 19.180$

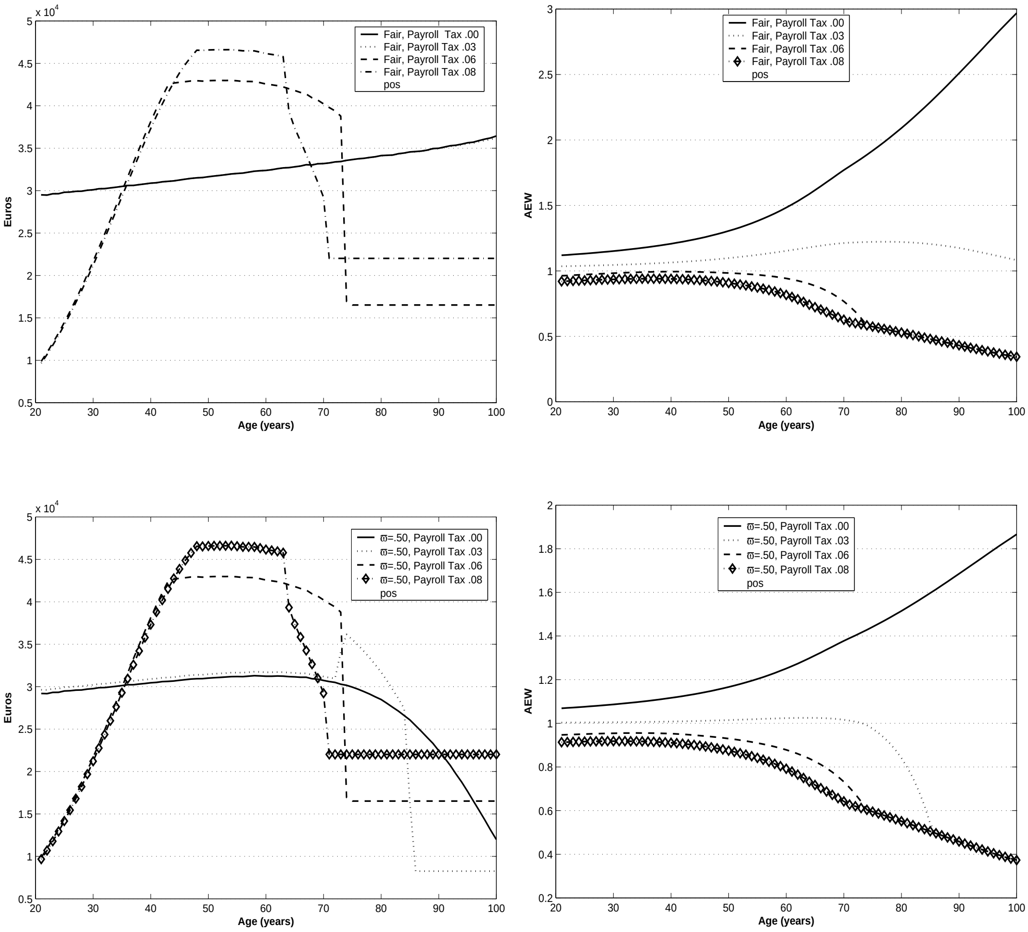


Table 3.19: Wealth by Age: Cohort 1960 (Men). Case $J = 65$, $\gamma = 2$, and $w = 17.179$

Age	Fair				$\varpi = .50$			
	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-83.965	-83.956	11.574	15.432	-83.481	-87.603	11.574	15.432
40	-69.438	-69.424	40.545	54.061	-65.543	-76.327	40.545	54.061
50	74.719	74.730	111.946	130.672	86.732	65.996	111.945	130.672
60	303.646	303.638	223.034	274.570	323.134	290.319	223.034	274.570
70	384.012	383.846	249.739	333.148	384.430	339.989	249.739	333.148
80	278.979	278.531	161.069	214.999	241.387	80.527	161.069	214.999
90	184.577	183.732	93.382	124.866	110.040	46.680	93.382	124.866
100	114.529	112.971	49.148	66.058	28.076	24.557	49.148	66.058

Figure 3.20: Consumption and A.E.W. by Age: Cohort 1960 (Men). $J = 65$, $\gamma = 2$, and $w = 17.179$

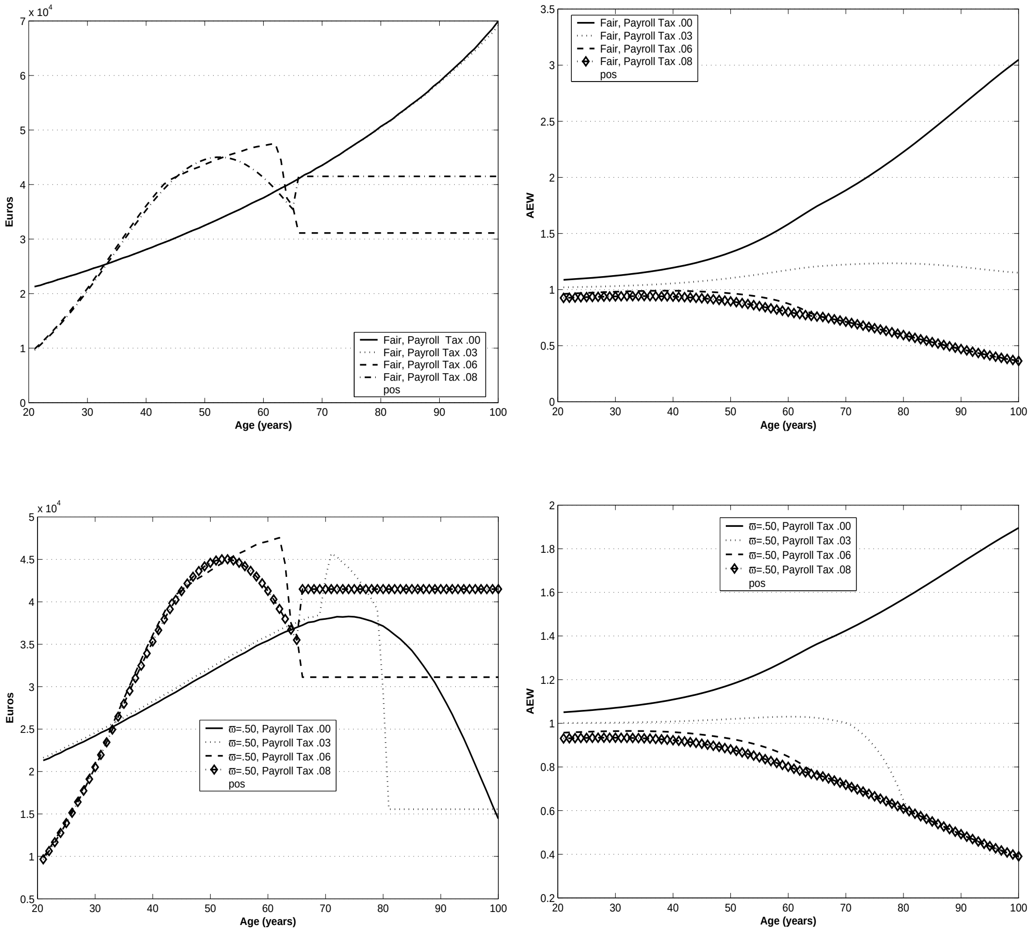


Table 3.20: Wealth by Age: Cohort 2000 (Men). Case $J = 65$, $\gamma = 2$, and $w = 17.179$

Age	Fair				$\varpi = .50$			
	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-74.530	-74.517	-74.504	7.232	-71.823	-74.252	-2.199	15.363
40	-42.999	-42.977	-42.955	39.971	-35.012	-41.380	28.544	54.876
50	127.318	127.371	127.371	207.424	144.290	131.978	194.087	181.911
60	383.541	383.559	383.577	455.720	409.774	389.943	437.628	362.738
70	464.774	464.692	464.609	523.872	482.643	458.283	418.706	319.558
80	341.009	340.770	340.532	315.589	329.737	305.199	141.569	189.029
90	220.571	220.122	219.674	107.788	168.388	97.841	80.843	108.192
100	128.069	127.235	40.736	53.611	50.103	19.872	40.212	54.209

Figure 3.21: Consumption and A.E.W. by Age: Cohort 2000 (Men). $J = 65$, $\gamma = 2$, and $w = 17.179$

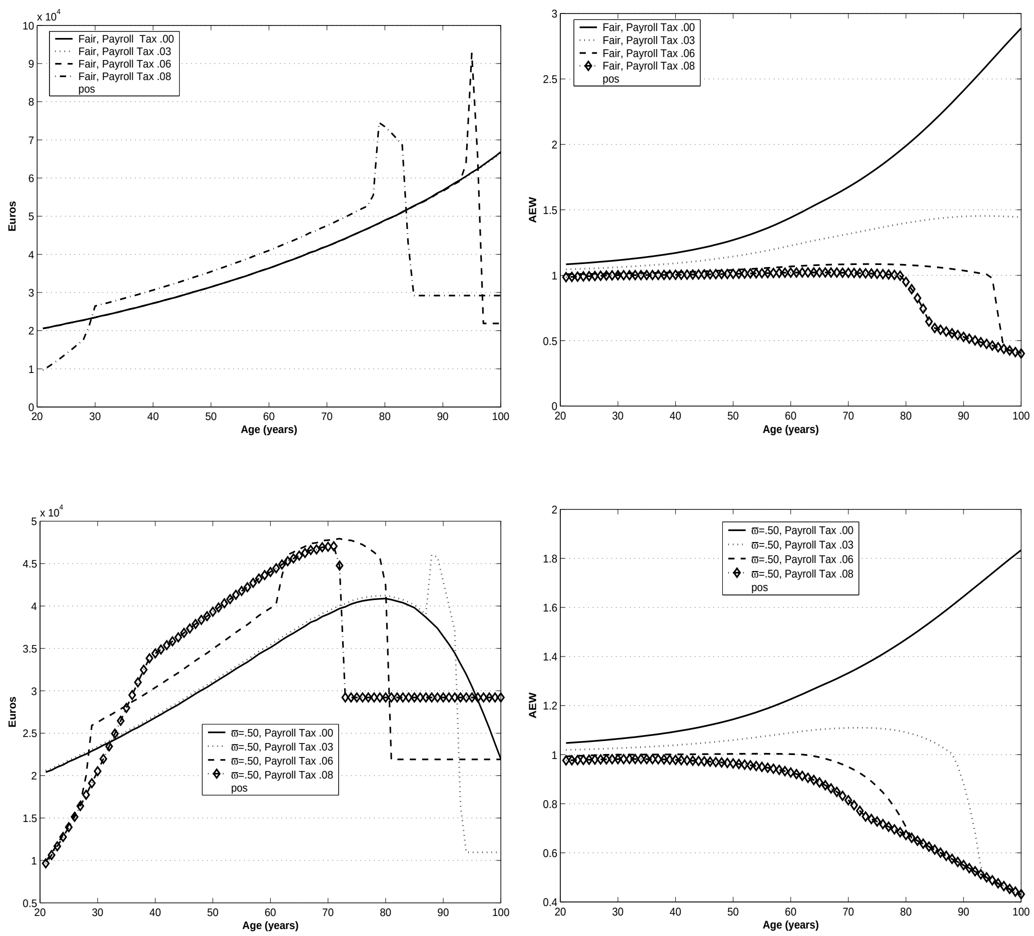


Table 3.21: Wealth by Age: Cohort 1960 (Men). Case $J = 70$, $\gamma = 2$, and $w = 17.179$

	Fair				$\varpi = .50$			
Age	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-82.845	5.684	11.368	15.157	-83.863	5.684	11.368	15.157
40	-78.476	18.159	39.266	52.355	-78.544	19.633	39.266	52.355
50	35.432	125.669	94.106	125.475	40.192	70.070	94.106	125.475
60	207.103	285.643	197.045	262.727	218.404	141.156	197.045	262.727
70	377.514	440.222	434.040	578.721	380.069	216.764	434.040	578.721
80	274.258	302.964	280.111	373.481	238.648	139.675	280.111	373.481
90	181.454	80.987	162.681	216.908	108.792	80.778	162.681	216.908
100	112.591	42.508	86.064	114.752	27.757	42.199	86.064	114.752

Figure 3.22: Consumption and A.E.W. by Age: Cohort 1960 (Men). $J = 70$, $\gamma = 2$, and $w = 17.179$

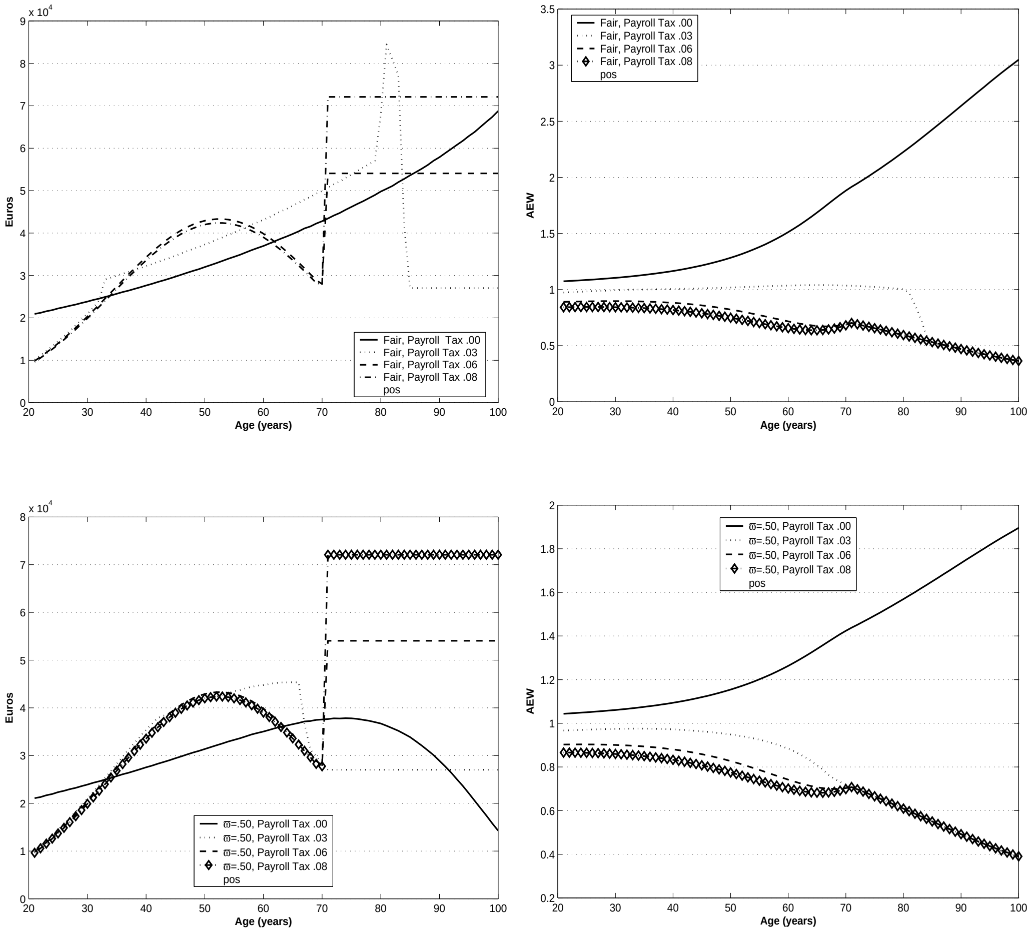
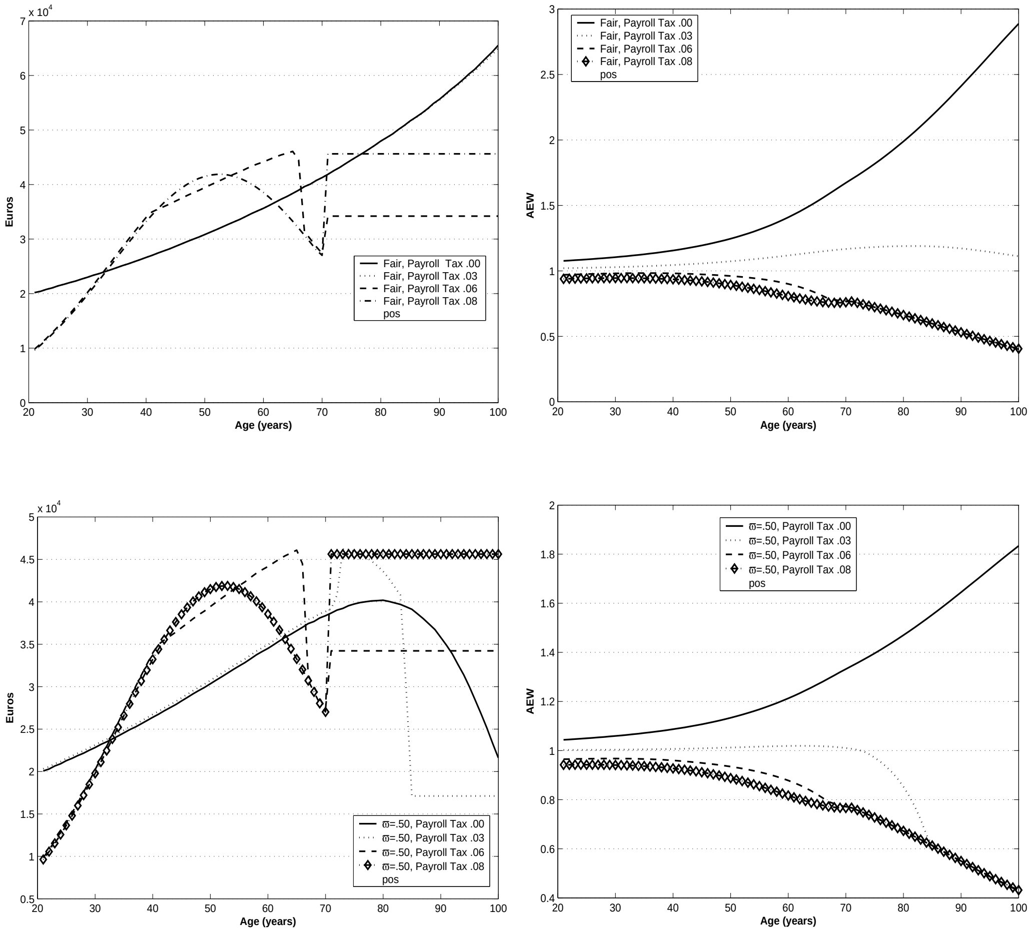


Table 3.22: WEALTH BY AGE: COHORT 2000 (MEN). CASE $J = 70$, $\gamma = 2$, AND $w = 17.179$

Age	Fair				$\varpi = .50$			
	0	3	6	8	0	3	6	8
20	0	6	12	15	0	6	12	15
30	-73.324	-73.314	11.276	15.034	-71.477	-74.799	11.276	15.034
40	-54.083	-54.067	38.425	51.216	-48.355	-57.091	38.425	51.216
50	81.022	81.036	119.265	119.304	93.862	76.773	119.265	119.394
60	276.509	276.506	223.591	234.501	298.097	269.414	223.591	234.501
70	455.388	455.337	337.427	450.221	474.409	435.534	337.427	450.221
80	334.122	333.807	221.146	295.334	324.112	190.385	221.146	295.334
90	216.116	215.453	126.312	169.115	165.516	63.320	126.312	169.115
100	125.483	124.202	62.870	84.862	49.248	31.677	62.870	84.862

Figure 3.23: CONSUMPTION AND A.E.W. BY AGE: COHORT 2000 (MEN). $J = 70$, $\gamma = 2$, AND $w = 17.179$



Chapter 4

Demography and Uncertainty in Economic Growth: An Application to Social Security

The aim of this chapter is to develop an OLG growth model, with realistic demography, that enables us to track each individual throughout her life-cycle. To do so, we use a longitudinal accounting to analyze the optimal allocation process of each individual, rather than the cross-sectional accounting, most frequently used. As a result, we find that there exist multiple steady states, e.g. the "*modified golden rule*" and the "*golden rule*" are feasible equilibria. Subsequently, we apply this model for studying the impact of a social security system on economic growth. Thus, we also find that both funded and unfunded systems can achieve the same steady state either the "*modified golden rule*", or the "*golden rule*". However, the dynamic transition of each social security system to the same steady state differs.

4.1 Introduction

In this chapter I propose to study how each social security system, either funded or unfunded, affects economic growth. A first approach to this issue was accomplished by Feldstein (1974). He pointed out by using a time-series study that the U.S. pay-as-you-go social security system depresses personal savings. This negative impact on capital stock, and thus on economic growth, is explained by the deadweight losses that create the collection of payroll taxes to finance retirement benefits. In particular, Boskin and Hurd (1978) show that an unfunded system distorts the labor supply and it diminishes savings over the life cycle due to differences in the marginal propensity to consume between aged and young people as well. However, the negative impact of the unfunded social security is far from being widely accepted, since many results rely on several restrictive assumptions about the behavior of the economic agent, the population structure, and even market features, among others.

The main advantage of this lack of consensus is the increasing research on the effect of social security on the economy. For example, by means of simulations we know that an unfunded social security, introduced into an economy composed of selfish individuals, crowds out the stock of physical capital, Auerbach (1987). But if individuals have altruistic feelings, Fuster et al. (2003) and Fuster (1999), the crowding out effect is reduced or eliminated. Also, there exists a crowding out when an actuarially fair social security is introduced into an economy with market failures in the provision of private annuities, Hubbard (1987) and Abel (1985). However, simulations of partial equilibrium models do not give a complete view of the effects produced by a social security in the long run, given that a complete understanding of this issue requires the knowledge of how every economic variable evolve as time goes by.

This chapter therefore develops an OLG growth model, with realistic demography, that enables us to track each individual throughout her life cycle. Thus, we extend the Cass-Koopmans-Ramsey model[1] by introducing a longitudinal account-

[1] Ramsey (1928), Cass (1965), and Koopmans (1965) analyze the evolution of the consumption

ing framework, instead of the cross-sectional accounting most frequently used. This approach to the problem has been already attempted by Bommier and Lee (2003).[2] Nonetheless, this model differs from Bommier and Lee (2003) in two aspects. On the one side, we present a longitudinal accounting framework that more easily allows economic interpretation. This is partly because we do not need to differentiate with respect to time and age. On the other side, this model is based on the Cass-Koopmans-Ramsey model instead of the Gale (1973) model. Other related papers such as Calvo and Obstfeld (1988) and Blanchard (1985) considered a dynamic continuous model with lifetime uncertainty as well. However, their models do not have a longitudinal accounting.

The results of this chapter show that a realistic demography coupled with the longitudinal accounting framework modifies the usual dynamic function of consumption per capita. Thus, in general, there will exist three kinds of steady state equilibria: i) *golden rule* equilibrium in which the interest rate equals the population growth rate as in Phelps (1966), ii) *modified golden-rule* equilibrium in which the interest rate equals the subjective discount factor plus a proportion of the population growth rate, and iii) a non-trivial equilibrium which depends on both economic and demographic variables. We also find that both funded and unfunded social security systems achieve the *golden rule* and the *modified golden-rule* equilibria. Therefore, contrary to previous research, an unfunded social security has no effect on saving rates and capital accumulation in the long run. However, there exists a temporal intergenerational problem along the transition to the steady state caused by the unfunded system. During this transition, there seems to be a crowding out effect on the stock of physical capital.

The model has five features. First, the economy is closed to both migration and investments from other economies. Second, there exits a productive firm that

and the stock of physical capital through the interaction between competitive firms and maximizing consumers. The main feature of these models is that the saving rate is an endogenous variable.

[2]This paper presents an excellent overview about the evolution of the use of demography in economic models.

combines labor and physical capital to produce a storable good, which can be either consumed or saved by each individual. Third, the population is composed of selfish people who face an uncertain length of life. Therefore, there are no intergenerational transfers. Fourth, each individual supplies her labor inelastically up to the age of retirement. This fact has a twofold implication. On the one hand, the population is divided into workers and retirees and, on the other hand, individuals save for a precautionary motive. Fifth, a selfish individual will prefer, following Yaari (1965), to purchase annuities, hence we assume that there exists a risk pooling that offers actuarially fair private annuities.

The remainder of this chapter proceeds as follows: Section 4.2 presents the longitudinal accounting framework applied to our population. We show how both cross-sectional and longitudinal frameworks are similar at any time. Some useful demographic functions and how they evolve over time will be also explained. Section 4.3 is devoted to developing the longitudinal accounting framework for the economic variables. Thus, the main aggregate and per capita functions, as well as their dynamic motion equations, are presented. Section 4.4 introduces the economic framework and analyzes the main results in an economy without social security. The impact of social security on economic growth is introduced in section 4.5. This Section is divided into two subsections in order to study the consequences to the economy of both funded and unfunded systems. Section 4.6 concludes. Section 4.7 contains an Appendix with the main proofs. Finally, some simulations complete the chapter.

4.2 Demographic Accounting Framework

A growth economic model with realistic demography is based on the structure of its population and its size over time. As a first step, we will set up the demographic background. Henceforth, we assume a "closed population" (no migration flows) that only changes through births B and deaths D. Thus, the population growth rate at time t, denoted as $n(t)$, corresponds to the crude birth rate minus the flow of deaths per capita at time t, $b(t) - d(t)$. Let us call those people who were born in year x "cohort x". The size of the cohort x at age s will be given by $\Omega_x(s)B(x)$, where $\Omega_x(s)$ is the probability that an individual who was born in year x will be alive at age s (see Definition 4.1 in the appendix). Therefore, the size of our population at time t, denoted as $P(t)$, is derived by adding up the size of every cohort alive at that time.

There are two different ways of calculating $P(t)$, either in a cross-sectional perspective or in a longitudinal perspective:

$$P(t) = \underbrace{\int_0^T \Omega_{t-s}(s)B(t-s)ds}_{Cross-Sectional} = \underbrace{\int_{t-T}^t \Omega_x(t-x)B(x)dx}_{Longitudinal} \tag{4.1}$$

where T is the maximum age, or longevity, of any individual. In this chapter, we use the longitudinal accounting, instead of the cross-sectional, because of the following two reasons. First, we do not need to differentiate with respect to time and age in order to derive the evolution of any socio-economic variable over time. With longitudinal accounting we simply need to differentiate with respect to time. Second, the longitudinal accounting enables us to simultaneously analyze the behavior of an individual both at age s and over her life cycle. Thus, given this accounting framework, we will proceed in each scenario by analyzing the individual life-cycle behavior, and then obtain the aggregate results.

Equation (4.1) is useful for aggregate functions but, for the sake of consistency with respect to previous growth models, we are also interested in per capita variables. The size per capita of the cohort x at time t is defined as $p_x(t-x)$:

$$p_x(t-x) = \frac{\Omega_x(t-x)B(x)}{P(t)} = \frac{\Omega_x(t-x)B(x)}{\int_{t-T}^t \Omega_x(t-\tau)B(\tau)d\tau}. \tag{4.2}$$

Then, equation (4.2) evolves over time according to the following law of motion:

$$\dot{p}_x(t-x) = -(\mu_x(t-x) + n(t))p_x(t-x), \text{ with } t \geq x, \tag{4.3}$$

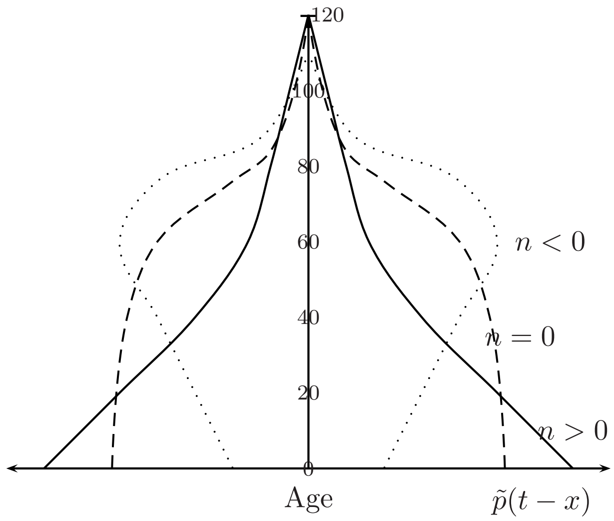
where $\mu_x(t-x)$ is the instantaneous mortality rate[3] of an individual of age $t-x$, being born at year x. In addition to equation (4.2), let us denote $\tilde{p}(t-x)$ as the

[3] μ is also called by demographers as "mortality hazard rate" and by actuaries as "force of mortality".

stable cohort x per capita at time t. From (4.3), $\tilde{p}(t-x)$ has the following expression:

$$\tilde{p}(t-x) = \Omega(t-x)b\mathrm{e}^{-n(t-x)},\ t \geq x. \tag{4.4}$$

Figure 4.1: Stable Population Structures



Previous models, such as Blanchard (1985)[4], assume that the population is stationary ($n(t) = 0, \forall t$) and that the instantaneous mortality rate is constant along the lifespan. On the contrary, and following Bommier and Lee (2003), we will consider a realistic mortality rate, since it improves our theoretical results and it does not introduce too much complexity into the model. Nonetheless, our main findings will be presented under a "stable population" structure. That is, the population grows at a rate equal to n, and so the size of any cohort per capita will remain constant over time.

Figure 4.1 above shows three hypothetical distributions of stable cohorts per capita according to age and population growth rate. Note that we only need to change the value of n in order to obtain a young age pyramid ($n > 0$, solid line) or an aging age pyramid ($n < 0$, dotted line).

[4] An individual with a constant instantaneous mortality rate has the same life expectancy at any age. According to this fact, her marginal propensity to consume will be constant over time.

4.3 Aggregate and Per Capita Functions

We will now develop an accounting framework to describe aggregate variables, per capita variables and an individual counterpart at time t, in which any realistic demography could be considered. In addition, we will also calculate how the previous variables evolve over time in order to determine their steady states.

We denote aggregate variables by uppercase letters Z, per capita variables by lowercase letters z, and an individual's variable by a lowercase letter followed by a subscript z_x.[5] The relation between an aggregate variable and individual counterpart is

$$Z(t) = \int_{t-T}^{t} z_x(t-x)\Omega_x(t-x)B(x)dx. \tag{4.5}$$

Equation (4.5) means that the aggregate value of any variable corresponds to the sum of the cohort's average values of that variable times the number of people within the cohort. For example, aggregate consumption is the sum of the average consumption of each cohort, multiplied by the number of people within the cohort. Similarly, by using equation (4.2) the per capita variable has the following formula:

$$z(t) = \frac{Z(t)}{P(t)} = \int_{t-T}^{t} z_x(t-x)p_x(t-x)dx. \tag{4.6}$$

The interpretation of equation (4.6) is simple. z is the average value of Z for our population at time t.

So far, we have shown our variable z only in a specific moment. We are now interested in knowing how equations (4.5) and (4.6) evolve. So, we will differentiate both equations with respect to time, which gives

$$\begin{aligned} \dot{z}(t) &= \int_{t-T}^{t} \dot{z}_x(t-x)p_x(t-x)dx - n(t)z(t) + \\ &\quad b(t)z_t(0) - \int_{t-T}^{t} \mu_x(t-x)z_x(t-x)p_x(t-x)dx, \end{aligned} \tag{4.7}$$

and

$$\dot{Z}(t) = P(t)(\dot{z}(t) + n(t)z(t)). \tag{4.8}$$

[5] The subscript denotes the year in which the individual was born.

To understand (4.7), we divide it into three terms. The first integral represents how z evolves for each current cohort. The second term represents that z decreases through dilution due to population growth and, finally, the third term depicts how z may change because of the increment produced by newborns and the reduction caused through deaths. This third part has not been taken into account before, except by Bommier and Lee (2003). However, it is crucial for analyzing growth models with overlapping generations, since young and old people do not necessarily allocate the same amount of their resources for a given variable z, as is generally considered. For example, according to (4.7), previous models are implicitly assuming that consumption of newborns, $b(t)c_t(0)$, equals the consumption of those people who have just died. That is,

$$b(t)c_t(0) = \int_{t-T}^{t} \mu_x(t-x)c_x(t-x)p_x(t-x)dx.$$

In order to better understand this inconsistency, we can imagine a stationary population structure. Under this demographic scenario, this equality only holds whenever the consumption of old people is the same as that of young people. Meanwhile, under an aging population, it holds only if old people consume less than young people. In sum, we can conclude that the latter equality does not seem to be realistic according to theoretical age-consumption profiles.

On the other hand, equation (4.8) suggests that, despite the fact that population growth diminishes per capita variables, aggregate variables are nonetheless positively affected by population growth.

4.4 Economic Framework

In our closed economy, there is only one firm that combines labor L and physical capital K to produce a single commodity or output, as a whole F. For simplicity, we assume that F is an homogeneous function of degree one, and that there is no technological progress. Thus, the production function takes the following modified intensive form

$$F(t) = P(t)f(k(t), l(t)) \equiv P(t)f(t) \tag{4.9}$$

where $k(t) = \frac{K(t)}{P(t)}$ is the physical capital per capita at time t, and $l(t) = \frac{L(t)}{P(t)}$ is the ratio workers-population at time t. Finally, f satisfies the conditions, $f \geq 0$, $f_k, f_l \geq 0$, $f_{kk}, f_{ll} \leq 0$, and $f_{kl} = f_{lk} \geq 0$, as well as the Inada conditions.

We assume that each individual supplies her labor force inelastically up to the age of J years old, in exchange of a salary w that corresponds to the marginal productivity of labor

$$w(t) = f_l(k(t), l(t)).$$

Afterwards, she decides to retire. Hence, the population at time t can be divided into workers $L(t)$ and retirees $L^r(t)$

$$P(t) = L(t) + L^r(t) = \underbrace{\int_{t-J}^{t} \Omega_x(t-x)B(x)dx}_{Workers} + \underbrace{\int_{t-T}^{t-J} \Omega_x(t-x)B(x)dx}_{Retirees}\,. \tag{4.10}$$

It is worth noting that by introducing retirees in our closed economy, there will be savings for the retirement motive. Furthermore, this assumption gives us insight into the consequences to the economy caused by a population with longer life expectancy. Therefore, it helps to determine the optimal retirement age, and even to study feasible policies for balancing this unfavorable circumstance with the social security system.

Individuals are assumed to have perfect foresight and do not have a bequest motive (selfish), so they only receive satisfaction through consumption c. In addition to these facts and with the intention of maximizing their utility, they constitute a pool for insuring their risk of mortality. Therefore, they purchase actuarially fair annuities in order to not outlive their resources in the case of survival beyond their life expectancy. This annuity contract has two important features. First, after paying the premium and if the individual survives at the end of the annuity contract, she will receive the marginal productivity of physical capital, or the safe interest rate r

$$r(t) = f_k(k(t), l(t)),$$

plus a risk premium μ contingent on her mortality risk. However, if she does not survive at the end of the annuity contract, she will lose her premium. Second, the purchase of annuities leads to newborns not receiving bequests. As a consequence, individuals start their life-cycle without physical capital. According to this economic scenario and following Yaari (1965), each individual belonging to any cohort x can be economically characterized by the following dynamic budget constraint:

$$\dot{k}_x(t-x) = (r(t) + \mu_x(t-x))k_x(t-x) + w_x(t-x) - c_x(t-x), \tag{4.11}$$

where

$$w_x(t-x) = \begin{cases} w(t) & if \quad 0 < t-x < J \\ 0 & if \quad t-x \geq J \end{cases} \tag{4.12}$$

and

$$k_x(0) = 0. \tag{4.13}$$

This dynamic budget constraint shows two key features. On the one hand, the greater physical capital rate of return is due to the fact that the individual purchases annuities. In particular, $\mu_x(t-x)k_x(t-x)$ corresponds to the proportion of wealth transferred from people within the cohort x who are recently deceased. On the other hand, by aggregating every individual within the economy we are able to see that lifetime uncertainty does not affect the stock of physical capital K in the short run. However, it does affect the future stock of physical capital in the long run through consumption. Consequently, we also need to study how consumption evolves over time.

If an individual's preferences can be represented by a CRRA utility function and every individual maximizes their lifetime utility, then the dynamic of consumption will be

$$\dot{c}_x(t-x) = \frac{r(t)-\delta}{\gamma} c_x(t-x) \text{ for all } t \geq x, \tag{4.14}$$

where $\gamma > 0$ is the constant risk aversion coefficient, and δ is the subjective discount parameter. Equation (4.14), known as the *Euler Equation*, implies that any individual of the cohort x will consume more (resp. less) at some date $t + dt$ whenever the market interest rate is greater (resp. lower) than her subjective discount factor.[6]

[6]Equation (4.14) holds at $t + dt$ if, and only if, we assume that individuals have perfect foresight about their future earnings.

Accordingly, as long as $r(t) \geq \delta$ we will expect that every present and future cohort will have a non decreasing consumption throughout her life-cycle.

The Dynamics of the Economy

Thus far, we have established the optimal individual behavior of any person within a cohort x, when she purchases actuarially fair private annuities and there is no social security system. Now we will use our longitudinal accounting framework to describe the evolution of this economy in terms of the aggregate variables: consumption per capita $c(t)$ and physical capital per capita $k(t)$ at any time t.

Our aim is to derive the stationary state, or stationary states, of this economy. To do so, we first proceed by deriving how physical capital and consumption evolve. Second, we will calculate the singular or multiple steady states of the economy.

We will begin by deriving the dynamic of physical capital per capita. Thus, we combine equation (4.7) with the optimal consumer's allocation process depicted by equations (4.11), (4.13), and (4.14). As a result, the dynamic of physical capital per capita at time t is

$$\dot{k}(t) = f(t) - c(t) - n(t)k(t). \tag{4.15}$$

This physical capital dynamic is the same as that in Solow (1956), Blanchard (1985), Bommier and Lee (2003), among many others. This is so, as we have already pointed out, because annuities transfer the wealth from deceased individuals to those individuals who are alive at the end of the annuity contract.

Similarly as in previous analyses, the latter equation also implies that in the steady state, or steady states, consumption per capita should be equal to

$$c = f - nk = (r - n)k + wl. \tag{4.16}$$

And as a consequence, by using equation (4.8), both aggregate physical capital and aggregate consumption will grow in a stationary state at a constant rate equal to n, independently of both the number of workers and the stock of physical capital per capita accumulated. That is,

$$\frac{\dot{K}}{K} = n = \frac{\dot{C}}{C}.$$

These previous results are all well-known. However, the problem arises when we study the dynamic of consumption per capita. Contrary to the variable physical capital per capita, we do not expect that consumption of those people who are born at time t, $c_t(0)b(t)$, equals the consumption of people recently deceased, $\phi_c(t)c(t)$. In fact, the former can be either greater, or equal, or lower than the latter.[7] By introducing (4.14) into (4.7), the dynamic of the consumption per capita is

$$\dot{c}(t) = \left(\frac{r(t)-\delta}{\gamma} - n(t)\right) c(t) + c_t(0)b(t) - \phi_c(t)c(t), \tag{4.17}$$

where $\phi_c(t)c(t)$ is equal to

$$\int_{t-T}^{t} \mu_x(t-x)c_x(t-x)p_x(t-x)dx.$$

Equation (4.17) says that the dynamic of consumption per capita is not well estimated, unless we subtract the consumption of people recently deceased, at the same time as we add the newborns' consumption. Unfortunately, because variables $c_t(0)$ and $c_x(t-x)$ depend on the interest rate $r(t)$, the stationary consumption per capita ($\dot{c} = 0$) is very difficult to obtain, perhaps even impossible, for a non-stable population. Consequently, we will assume henceforth that our population is stable. On the other hand, in steady states the dependence of t disappears. Thus, we can use the cross-sectional accounting, instead of the longitudinal accounting, in order to derive steady states. Thus, the next theorem gives the necessary conditions for a stationary economy under the cross-sectional accounting framework.

Theorem 4.1 *Let us assume a closed population that faces an uncertain length of life and grows at a constant rate equal to n. If private markets offer actuarially fair annuities, then consumption per capita and physical capital per capita will be stationary for any interest rate that satisfies either:*

$$r = \delta + \gamma n, \tag{4.18}$$

or

$$(r-n)\,k + wl = c(0)\int_0^T \mathrm{e}^{\frac{r-\delta}{\gamma}s}\tilde{p}(s)ds. \tag{4.19}$$

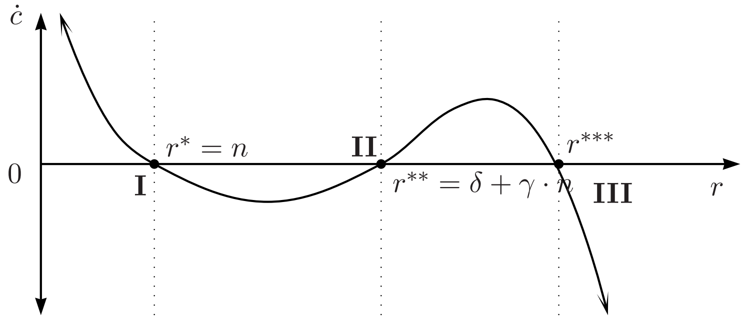
[7]This result is different from that of the variable physical capital mainly stems from the fact that consumption is not a stock variable.

Proof. See Appendix. ■

Theorem 4.1 claims that, given a constant population growth rate, there can be more than one feasible equilibrium point in the economy. We find that one of the multiple steady states corresponds to the consumption and physical capital per capita in which r is equal to $\delta+\gamma n$. This steady state is known as "*the modified golden rule*". We also find that whenever the equation (4.19) is satisfied, the *modified golden rule* is not the only steady state that this economy can reach. In fact, according to (4.19) there can be two additional steady states. One is a non-trivial steady state that not only depends on socio-demographic variables $\{J, T, \Omega, b,$ and $d\}$, but also on economic variables $\{r, w, \gamma$ and $\delta\}$ (point **III** in Figure 4.2). The other is a steady state that corresponds to the "*golden rule*" condition, $r = n$ or point **I** in Figure 4.2.

The number of steady states in an annuitized economy depends on the population growth rate. While a growing population (see figure 4.2) presents three steady states, a decreasing population has either two or only one steady states depending on the value of n. Thus, an economy with a population growth rate within the interval $\left(-\frac{\delta}{\gamma}, 0\right)$ has two steady states. This economy cannot reach the *golden rule* equilibrium. On the other hand, if the population growth rate is lower than $-\frac{\delta}{\gamma}$, this economy will only have one steady state. In this particular case, the *modified golden rule* condition will not be attainable.

Figure 4.2: The Dynamic of Consumption per Capita for an Annuitized Economy with Perfect Foresight Agents. Case: $n, \delta > 0$ and $\gamma > 1$.



Analyzing the phase diagram and the eigenvalues associated to each equilibrium point, we can study whether each equilibrium is an attractor point or not. Note that our dynamic system is not linear, hence we will proceed linearizing equations (4.15) and (4.17) in the neighborhood of each steady state which satisfies theorem 4.1. The linearization gives[8]

$$\begin{pmatrix} \dot{k}(t) \\ \dot{c}(t) \end{pmatrix} \approx \begin{pmatrix} r(k^*) - n & -1 \\ \frac{\partial \dot{c}}{\partial k}\Big|_{c^*,k^*} & \frac{r(k^*)-\delta}{\gamma} - n \end{pmatrix} \cdot \begin{pmatrix} k(t) - k^* \\ c(t) - c^* \end{pmatrix}. \tag{4.20}$$

The roots of the characteristic equation associated with this Jacobian matrix are:

$$\lambda_{1,2} = \left(1 + \frac{1}{\gamma}\right) \frac{r(k^*)}{2} - \frac{\delta}{2\gamma} - n \pm \sqrt{\left(\left(1 - \frac{1}{\gamma}\right) \frac{r(k^*)}{2} + \frac{\delta}{2\gamma}\right)^2 - \frac{\partial \dot{c}}{\partial k}\Big|_{c^*,k^*}} \tag{4.21}$$

Equation (4.21) shows that the eigenvalues can take not only both positive and negative values, but also complex ones. Looking at figure 4.2 we see that $\frac{\partial \dot{c}}{\partial k}$ is positive in the neighborhood of points I and III, and yet the slope at each point is different. Through some simulations we have found that if the economy has three equilibria, the middle steady state will be a saddle point, and the remaining equilibria will be one stable and one unstable focus. Therefore, this economy can present four different paths: i) a spiral sink, ii) a saddle path, iii) an unstable spiral, and finally iv) a limit cycle between the attractor and the repeller equilibrium points. Three of the four paths describe oscillated movements. Thus, it is more likely to see economic cycles, even with perfect foresight agents. On the other hand, an economy with two equilibria has an unstable focus and a saddle point. This economy will only have one, instead of two, attractor equilibrium. Finally, an economy with a population growth rate lower than $-\frac{\delta}{\gamma}$ has a repeller equilibrium point.

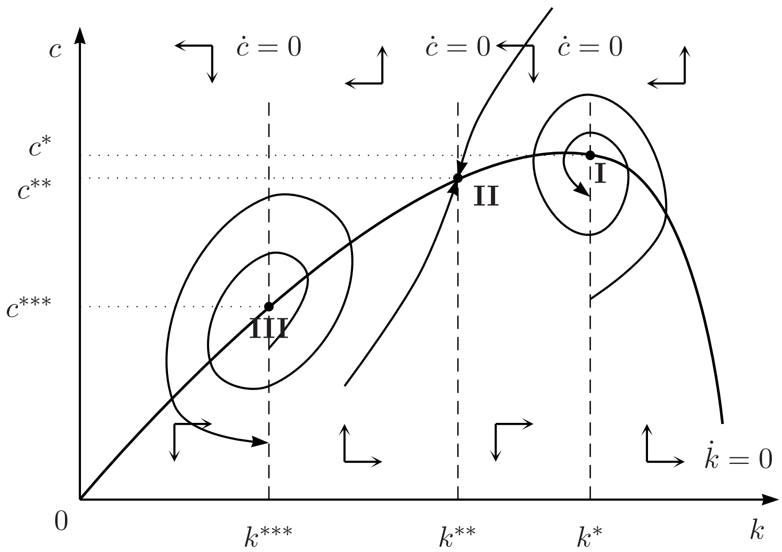
Figure 4.3 below shows the phase diagram of an annuitized economy with a growing population, and a constant risk aversion coefficient greater than one, and which is associated with Figure 4.2. According to the population growth rate, there

[8]Where the partial derivative of $\dot{c}$ with respect to k in the neighborhood of (k^*, c^*) is

$$\frac{\partial}{\partial k}\left[\left(\frac{r(k)-\delta}{\gamma} - n\right) \cdot \left((r(k) - n)k + w(k)l - c(0)b \int_0^T e^{\left(\frac{r(k)-\delta}{\gamma} - n\right)s} \Omega(s) ds\right)\right]_{c^*,k^*}.$$

are three steady states (**I**, **II** and **III**). Point **I** corresponds to the golden rule interest rate, and point **II** is the modified golden rule interest rate. Obviously, the steady state that this economy can reach without social security will depend on initial values of consumption and physical capital per capita. It is worth noting that so long as the economy moves towards point **I** or point **III**, many economic variables (e.g. the wage, the interest rates, the consumption per capita, and the physical capital per capita) will have an oscillated motion.

Figure 4.3: The Phase Diagram for an Annuitized Economy with Perfect Foresight Agents. Case: $n, \delta > 0$ and $\gamma > 1$.



After analyzing every feasible steady state, it is clear that whenever the economy reaches an equilibrium, neither the interest rate, nor the salary is generally modified. This is true except when the steady state is located in point **III**. However, and although both factor prices do not change when any steady state is attained, differences among countries may still appear due to demographic factors. For example, we find that both the consumption and the stock of physical capital per capita are negatively affected by the proportion of retirees. If we analyze two stationary economies with the same population growth but different life expectancies after the date of retirement, then the economy with the greater life expectancy will have a lower consumption and physical capital per capita than the economy with the lower life expectancy. This is because individuals in both economies earn the same wage and have the same interest rate. However, individuals with a greater life expectancy need to consume less in order to spread their consumption over their life-cycle, as Levhari and Mirman (1977) have already pointed out.[9]

4.5 Impact of Social Security in Economic Growth

The social security system has been widely criticized because it causes a crowding out effect on both the stock of physical capital and labor market or, equivalently, it has a negative impact on individuals' decision about retirement and saving. These conclusions were pointed out by Feldstein (1974) and Boskin and Hurd (1978), among many others. In addition, the unfunded method of financing Social Security has been identified as the worst method when one anticipates an aging population. This demographic scenario undoubtedly yields either a higher payroll tax for workers, reducing their disposable income, or a lower benefit for retirees, or a combination of both. However, there are many other effects that have not been extensively studied in a growth model so far. For example, how Social Security affects the steady state when realistic demography is taken into account has not been addressed. A pioneer work in this field is Bommier and Lee (2003). They find that a closed economy with capital and a social security system is able to reach a steady state that is either "*golden rule*" or "*balanced*".[10] Following the idea of multiple equilibria, we also show that both a funded and an unfunded social security can reach a steady state that is either "*golden rule*", or "*modified golden rule*". Nonetheless, both social security systems are not completely equivalent even with a stable population, since they approach the steady state following different trajectories. According to this result, the crowding out effect produced by an unfunded social security does not continue forever, suggesting that it is just a temporal intergenerational problem.

[9]As a consequence, this decrement in consumption can be balanced by either extending the age of retirement, or increasing the productivity.

[10]They extended the OLG growth model of Gale (1973) by introducing a productive firm.

4.5.1 ***Funded Social Security***

To start the analysis we first consider an actuarially fair and funded social security system. In order to focus on the study of the pension system, we leave aside common social security expenditures such as health care, unemployment, etc. Consequently, we assume that Social Security only levies a payroll tax τ on gross earnings, in exchange of a future benefit b when people retire. The equation (4.12) can be rewritten as the following piecewise function:

$$w_x(t-x) = \begin{cases} (1-\tau)w(t) & if \quad 0 < t-x < J \\ b_x(t-x) & if \quad t-x \geq J \end{cases}$$

where

$$b_x(t-x) = b = \tau \frac{\int_0^J w(x+j)\Omega_x(j)\mathrm{e}^{-\int_x^{x+j} r(p)dp} dj}{\int_J^T \Omega_x(j)\mathrm{e}^{-\int_x^{x+j} r(p)dp} dj}. \tag{4.22}$$

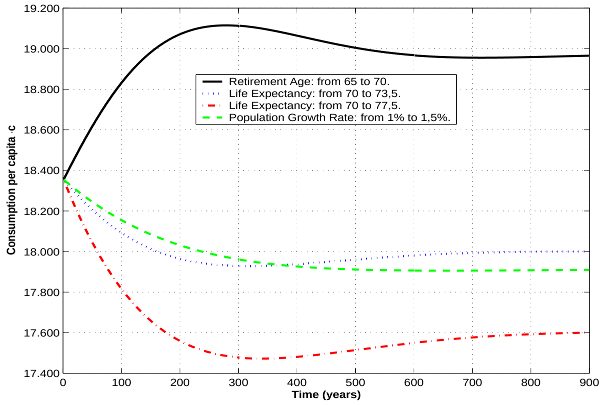
Equation (4.22) shows that our individual receives a flat pension benefit at retirement. b has the feature of being actuarially fair, and so the retiree receives the same return as if she would have invested her savings in the private market. However, this fact does not necessarily mean that a funded system is not affected by the population growth rate. According to this framework, the amount of money received through pension benefits will depend on both the interest rate and the survival probability. Thus, if b depends on the interest rate which, at the same time, is a function of the population growth rate, then the funded system will be affected by n as well. In order to show this influence and its implications for the economy, we have included the Figure 4.4 below.

In this figure we present four economic transitions, from a golden rule equilibrium to another golden equilibrium for different demographic changes. Subsequently, we will compare these results with the analogous case showed of the unfunded system shown in Figure 4.5.

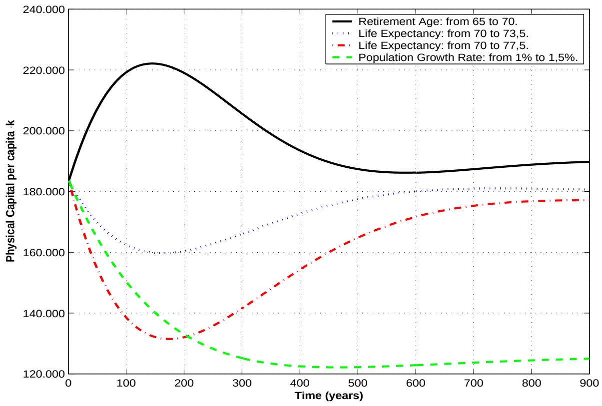
First, we should note that a funded social security does not modify the individual allocation process depicted by equations from (4.11) to (4.14).[11] Therefore, the

[11] Given this economic framework, both funded and unfunded social security systems only affect the dynamic of consumption per capita $\dot{c}$ on the initial consumption, $c_x(0)$. Using algebra we can

Figure 4.4: ECONOMIC TRANSITIONS BETWEEN EQUILIBRIA POINTS: FUNDED SOCIAL SECURITY



(a) Consumption per capita $\dot{c}(t)$



(b) Physical Capital per capita $\dot{k}(t)$

Note. The analysis assumes a CES production function $A\left(\theta k^{\alpha}+(1-\theta)l^{\alpha}\right)^{\frac{1}{\alpha}}$, where the parameters used are $\{A=2093, \theta=\frac{1}{3}, \alpha=-0.131855\}$, and $\delta=0.01$, and $\gamma=1$. The other parameters are reported in the figure.

---

easily prove that under a funded system, the initial consumption does not change. Thus,

$$c_x(0)=\frac{(1-\tau)\int_0^J w(x+j)\Omega_x(j)\mathrm{e}^{-\int_x^{x+j} r(p)dp}dj+b\int_J^T \Omega_x(j)\mathrm{e}^{-\int_x^{x+j} r(p)dp}dj}{\int_0^T \Omega_x(j)\mathrm{e}^{-(1-\frac{1}{\gamma})\int_x^{x+j} r(p)dp}\mathrm{e}^{-\frac{\delta}{\gamma}j}dj}.$$

Note that if we substitute b by equation (4.22), the initial consumption will be the same as in the case of not having a Social Security. That is,

$$c_x(0)=\frac{\int_0^J w(x+j)\Omega_x(j)\mathrm{e}^{-\int_x^{x+j} r(p)dp}dj}{\int_0^T \Omega_x(j)\mathrm{e}^{-(1-\frac{1}{\gamma})\int_x^{x+j} r(p)dp}\mathrm{e}^{-\frac{\delta}{\gamma}j}dj}.$$

steady states of this economy are the same as those equilibria obtained in the benchmark economy explained in Section 4.4. Secondly, we have chosen the golden rule equilibrium in this Figure because, on the one hand, it is the steady state in which the population maximizes its welfare, and, on the other hand, it is an attractor point which makes easier future comparisons.[12]

Figure 4.4 shows that an economy with a funded social security is negatively affected by an increase in both the life expectancy (dotted line, and dotted-dashed line) and the population growth rate (dashed line). In contrast, delaying the mandatory age of retirement from 65 years old to 70 years old raises both consumption and physical capital per capita. Therefore, the latter policy seems to be the most convenient in the case of having a growing population with an increasing life expectancy. We can also observe from Figure 4.4 that a funded system does not contain many cyclical movements, since the number of cycles depend on the number of oscillations. On the other hand, and contrary to the unfunded system, these economic transitions are independent of the payroll tax levied even though both consumption and physical capital per capita are modified at an individual level. This circumstance plays an important role in the next subsection through the number of oscillations.

4.5.2 *Unfunded Social Security*

Under an unfunded social security system, current workers support the pension benefits received by retirees, redistributing income over multiple generations. The pension benefits at time t in this case is:

$$b_x(t-x) = \tau w(t) \frac{\int_{t-J}^{t} p_x(t-x)dx}{\int_{t-T}^{t-T} p_x(t-x)dx} = \tau w(t) \frac{l(t)}{1-l(t)} \tag{4.23}$$

This pension benefit is a function of both the population structure through $l(t)$ and the current economic status through $w(t)$. Consequently, an unfunded social security system is more affected than a funded system by the population structure at any time. Now, comparing the pension benefit received from a pure funded social

[12]Unfortunately, the time passed in reaching another steady state is too long for a stable population. By contrast, an economy with a demographic transition has faster movements.

security, see equation (4.22), with that of an unfunded social security, we can observe under steady state conditions that the following relationship is satisfied:

$$\begin{cases} \tau w \frac{l}{1-l} < \tau w \frac{\int_0^J \mathrm{e}^{-rs}\Omega(s)ds}{\int_J^T \mathrm{e}^{-rs}\Omega(s)ds} & \text{if } r > n \\ \\ \tau w \frac{l}{1-l} > \tau w \frac{\int_0^J \mathrm{e}^{-rs}\Omega(s)ds}{\int_J^T \mathrm{e}^{-rs}\Omega(s)ds} & \text{if } r < n \end{cases}. \tag{4.24}$$

The importance of this relationship for our analysis is twofold. One, it helps to give insight into which system yields a greater pension benefit for a given payroll tax τ. Thus, once we know the steady state, we can specify which social security system yields a greater welfare. Second, because an unfunded social security does not offer the same return as private markets, initial consumption is not the same as that in a funded system. Therefore, we may wonder whether an unfunded system affects the steady states that this economy reaches.

It is easy to see from theorem 4.1 that the modified golden rule is independent of the sort of social security system that an economy has. However, equation (4.18) is not the same, due to the initial consumption. In particular, $c(0)$ now has the following expression

$$\frac{(1-\tau)w\int_0^J \Omega(s)\mathrm{e}^{-rs}ds + \tau w \frac{l}{1-l}\int_J^T \Omega(s)\mathrm{e}^{-rs}ds}{\int_0^T \mathrm{e}^{-\left(\left(1-\frac{1}{\gamma}\right)r+\frac{\delta}{\gamma}\right)s}\Omega(s)ds}. \tag{4.25}$$

Equation (4.25) shows that regarding (4.24) $c(0)$ is greater (resp. lower) under an unfunded system than $c(0)$ under a funded system whenever $r <$ (resp. $>$) n. Therefore, we can claim that an unfunded social security leaves the population better (resp. worse) off whenever an stationary economy will have an interest rate lower (resp. greater) than the population growth rate. Following this reasoning, we find that

Proposition 4.1 *Under the golden rule equilibrium both funded and unfunded social security systems yield the same welfare for every cohort.*

Proof. See Appendix. ■

Thus, if the economy is in the golden rule equilibrium, there will be no interest in switching from one social security system to another.[13] That is, every individual will have the same consumption trajectory under both social security systems. In addition to proposition 4.1, we also find that

Proposition 4.2 *Both funded and unfunded social security systems can reach either the modified golden rule, or the golden rule equilibrium.*

The proof is obvious from the observation of theorem 4.1 and proposition 4.1.

Theorem 4.1 gives three equilibria for a growing population. In fact, it is not included in proposition 4.2 because both systems do not attain the same non-trivial steady states showed in figure 4.3. On the other hand, it is important to note that insofar as $r = \delta + \gamma n$ will be lower than n, an unfunded system will be more desirable than a funded system, and vice versa. Therefore, the risk aversion coefficient γ and the subjective discount factor δ seem to be crucial variables for determining the optimal social security system under this model.

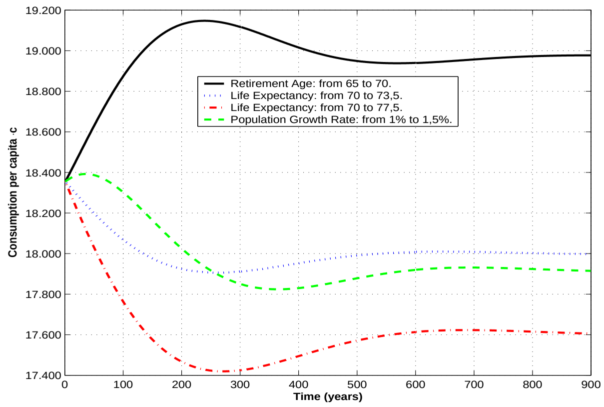
According to proposition 4.2, the non-trivial steady state is the only equilibrium point that changes. However, it is not the optimal equilibrium, since it does not maximize the aggregate consumption as the golden rule does. Given this circumstance, we will focus henceforth on the golden rule equilibrium.

We have pointed out that an unfunded social security modifies the dynamic of consumption per capita. Hence, the eigenvalues reported in equation (4.21) differ from the funded system eigenvalues because of the function $\frac{\partial \dot{c}}{\partial k}\big|_{k^*,c^*}$. If we consider that the golden rule is a spiral sink, we can assure that the speed of convergence from an initial state (k_0, c_0) to the steady state (k^*, c^*) is the same for both systems. In other words, in both systems the real point of the eigenvalues $Re(\lambda)$ is equal. However, the imaginary part of the eigenvalues $Im(\lambda)$ for both systems are not equal. As a consequence, an unfunded system moves towards the equilibrium describing a different trajectory then a funded system. Figure 4.5 shows the economic transitions from a golden rule equilibrium to another under the same demographic changes

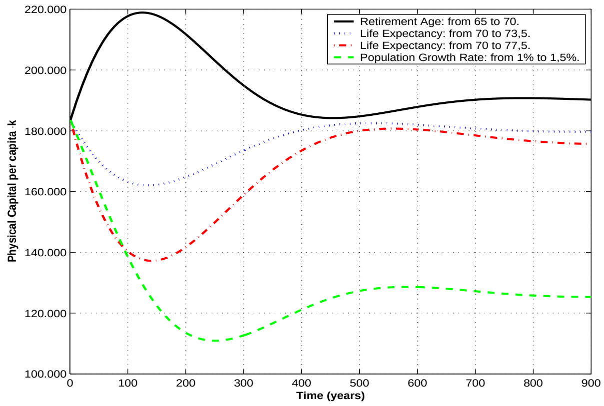
[13] Of course, the population growth rate should be positive due to the impossibility that the interest rate will be negative.

introduced in figure 4.4.

Figure 4.5: Economic Transitions between Equilibria Points: Unfunded Social Security ($\tau = 0.08$)



(a) Consumption per capita $\dot{c}(t)$



(b) Physical Capital per capita $\dot{k}(t)$

Note. The analysis assumes a CES production function $A\left(\theta k^{\alpha} + (1-\theta)l^{\alpha}\right)^{\frac{1}{\alpha}}$, where the parameters used are $\{A = 2093, \theta = \frac{1}{3}, \alpha = -0.131855\}$, and $\delta = 0.01$, and $\gamma = 1$. The other parameters are reported in the figure.

Looking at figures 4.4 and 4.5 we can see how both systems reach the same equilibria, although the economic trajectories vary between social security systems. In particular, an unfunded system has more oscillations than a funded system. The number of oscillations grows as the payroll tax raises. Furthermore, an important finding shown in Figure 4.5 is that a negative demographic impact affects an unfunded system more heavily than a funded system. Also, a positive demographic impact to the economy causes the physical capital per capita to grow at the beginning faster under a funded system than under an unfunded one. Therefore, given a demographic change, the economic transition between two golden rule equilibria is a temporal intergenerational problem, which is better off under a funded system only at the beginning of the transition.

4.6 Conclusions

This chapter shows that both the introduction of realistic demography and a longitudinal accounting have very important effects in economic growth models. Under a Cass-Koopmans-Ramsey model, we find that there exist multiple equilibria. For example, for a positive population growth rate, there exist three equilibria: i) *golden rule*, ii) *modified golden-rule*, and iii) a non-trivial equilibrium which depends on both economic and demographic variables. We also find that the number of steady states is an increasing function of the population growth rate, which goes from a minimum of one to a maximum of three. Further, the equilibrium paths not only can be a saddle path, but also a spiral sink and an unstable spiral. Therefore, this model with an usual production function leads to cyclical movements even with perfect foresight agents.

Afterwards, we have applied the model to study the impact of social security on economic growth. We find that both funded and unfunded social security systems can achieve either the *modified golden-rule* or the *golden rule* equilibrium. However, only under the golden rule equilibrium do both systems yield the same welfare for every individual. This first result suggests that Social Security does not crowd out the stock of physical capital in the long run, although it does before the economy reaches the steady state. Second, whether an unfunded system is preferred than a funded system depends on many demographic and economic variables than must to be further researched.

4.7 Appendix

Definition 4.2 *Let $\Omega \in \mathcal{C}^2([0,T))$ denote the rational discount function. That is, the probability that the consumer will be alive at age s. Ω has the following properties:*

1. $\Omega(0) = 1$.

2. $\lim_{s\to T} \Omega(s) = 0$.

3. $0 < \Omega(s) \leq 1$.

4. $\Omega(\tau) < \Omega(s) \Leftrightarrow \tau > s$.

5. $-\frac{\frac{d}{ds}\Omega(s)}{\Omega(s)} = \mu(s) > 0$.

   where $\mu \in \mathcal{C}^\infty([0,T))$ is the instantaneous mortality rate.

6. *The mortality hazard rate μ is an increasing function of age*[14]

   $\frac{d}{ds}\mu(s) \geq 0$, $\forall s \in [0,T)$.

7. *In particular, the probability of being alive at age x is given by the following mapping:*

$$\begin{aligned} \Omega : [0,T) &\rightarrow (0,1] \\ s &\mapsto \Omega(s) = \mathrm{e}^{-\int_0^s \mu(\tau)d\tau} \end{aligned}$$

Proof of Theorem 4.1. In order to prove theorem 4.1, we will proceed in two steps. First, we will derive the two necessary conditions for a steady state economy. Second, we will show how the golden rule interest rate also satisfies this theorem.

We use the cross-sectional accounting framework because we have assumed that the population is stable. By hypothesis, the optimal consumption of any individual at age s can be depicted by

$$c(s) = c(0)\mathrm{e}^{\frac{r-\delta}{\gamma}s}, \forall s \in [0,T).$$

[14]In general, this is not true at young ages, although it is assumed for the sake of simplicity.

Thus, by plugging the consumption trajectory into equation (4.17), the dynamic of consumption per capita is

$$\dot{c} = \left(\frac{r-\delta}{\gamma} - n\right) c + c(0)b\left(1 - \int_0^T \mathrm{e}^{\left(\frac{r-\delta}{\gamma}-n\right)s}\mu(s)\Omega(s)ds\right).$$

Using the definition 4.2, property 5, we know that $-\mu(s)\Omega(s)$ can be substituted by $\frac{d\Omega(s)}{ds}$. Now, recalling u as $\mathrm{e}^{\left(\frac{r-\delta}{\gamma}-n\right)s}$ and dv as $d\Omega(s)$, and applying the rule $u \cdot v|_0^T - \int_0^T v \cdot du$, we get

$$\dot{c} = \left(\frac{r-\delta}{\gamma} - n\right)\left(c - c(0)b\int_0^T \mathrm{e}^{\left(\frac{r-\delta}{\gamma}-n\right)s}\Omega(s)ds\right).$$

A steady state economy should satisfy that $\dot{k} = 0 \Leftrightarrow c = f - nk$. Furthermore, given that f is a homogeneous function of degree one, we know that the latter condition can be rewritten as $c = (r-n)k + wl$. Therefore, a necessary condition for an steady state economy must satisfy that both $\dot{c} = 0$ and $\dot{k} = 0$, or equivalently:

$$r = \delta + \gamma n,$$

or

$$(r-n)k + wl = c(0)b\int_0^T \mathrm{e}^{\left(\frac{r-\delta}{\gamma}-n\right)s}\Omega(s)ds.$$

Up to this point we have proven that an economy with an interest rate that satisfies either equation (4.18) or (4.19) is under a steady state. From equation (4.18) we immediately know that an interest rate is the modified golden rule. However, we have not yet specified an interest rate contained in the equation (4.19). In order to show that the golden rule is a possibility, we first need to evaluate the initial consumption. Thus,

$$c(0) = \frac{w\int_0^J \Omega(s)\mathrm{e}^{-rs}ds}{\int_0^T \mathrm{e}^{-\left(\left(1-\frac{1}{\gamma}\right)r + \frac{\delta}{\gamma}\right)s}\Omega(s)ds}.$$

Plugging the initial consumption into the equation (4.19), we find that

$$(r-n)k + wl = \frac{w\int_0^J \Omega(s)b\mathrm{e}^{-rs}ds\int_0^T \mathrm{e}^{\left(\frac{r-\delta}{\gamma}-n\right)s}\Omega(s)ds}{\int_0^T \mathrm{e}^{-\left(\left(1-\frac{1}{\gamma}\right)r + \frac{\delta}{\gamma}\right)s}\Omega(s)ds}.$$

The substitution of r by the golden rule condition n gives the following relationship:

$$l = \int_0^J \Omega(s)b\mathrm{e}^{-rs}ds$$

which is exactly the proportion of workers per capita under an stable population. Therefore, $r = n$ satisfies the equation (4.19).

■

Proof of Proposition 4.1. Given that private markets offer actuarially fair annuities and the population structure is stable over time, we only need to demonstrate that the initial consumption under both systems is the same whenever $r = n$. Thus, calling equation (4.25)

$$c(0) = \frac{(1-\tau)w\int_0^J \Omega(s)\mathrm{e}^{-rs}ds + \tau w\frac{l}{1-l}\int_J^T \Omega(s)\mathrm{e}^{-rs}ds}{\int_0^T \mathrm{e}^{-\left(\left(1-\frac{1}{\gamma}\right)r+\frac{\delta}{\gamma}\right)s}\Omega(s)ds}$$

and rearranging its numerator, gives

$$c(0) = \frac{w\int_0^J \Omega(s)\mathrm{e}^{-rs}ds + \tau w\left(\frac{l}{1-l}\int_J^T \Omega(s)\mathrm{e}^{-rs}ds - \int_0^J \Omega(s)\mathrm{e}^{-rs}ds\right)}{\int_0^T \mathrm{e}^{-\left(\left(1-\frac{1}{\gamma}\right)r+\frac{\delta}{\gamma}\right)s}\Omega(s)ds}.$$

Multiplying and dividing by b in the parenthesis located in the numerator, we have that

$$c(0) = \frac{w\int_0^J \Omega(s)\mathrm{e}^{-rs}ds + \frac{\tau w}{b}\left(\frac{l}{1-l}\int_J^T \Omega(s)b\mathrm{e}^{-rs}ds - \int_0^J \Omega(s)b\mathrm{e}^{-rs}ds\right)}{\int_0^T \mathrm{e}^{-\left(\left(1-\frac{1}{\gamma}\right)r+\frac{\delta}{\gamma}\right)s}\Omega(s)ds}.$$

Now, substituting r by n, and using equations (4.4), and (4.10), we obtain that

$$c(0) = \frac{w\int_0^J \Omega(s)\mathrm{e}^{-ns}ds}{\int_0^T \mathrm{e}^{-\left(\left(1-\frac{1}{\gamma}\right)n+\frac{\delta}{\gamma}\right)s}\Omega(s)ds}.$$

This result is the same as the initial consumption of a funded social security.

■

4.8 Simulations

We have studied in Chapters 2 and 3 the allocation process of an individual, and its consequences to certain economic variables such as wealth, consumption, and the demand for private annuities. Now, we are interested in studying, through various simulations, how an economy populated by perfect foresighted agents converges to the golden rule equilibrium.

In this chapter we have showed that an economy either with a funded or with an unfunded social security system achieves the golden rule equilibrium. Furthermore, the economy as a whole will prefer to stay at this steady state, since the consumption per capita is maximized. Therefore, we will focus in this subsection on the golden rule equilibrium and its properties.

We have divided this subsection according to Table 4.2 below. The first figure shows the economic transitions from the initial state to the golden rule, when the population has the survival probability of the cohort born in 1940 and the mandatory age of retirement is 65 years old. The next figure shows the economic transitions of the previous population, but now the mandatory age of retirement is 70 years old. The rest of the figures continue with different survival probabilities.

The majority of figures have six paths. They represent the economic transitions of a funded and an unfunded social security according to different payroll taxes, as well as different risk aversion coefficients. In particular, we simulate two payroll taxes $\{.075, .15\}$ of an unfunded social security. Additionally, a payroll tax equal to zero represents a funded social security system, since this system does not affect the allocation process of an individual. In addition to the social security, we have also assumed two risk aversion coefficients $\{.75, 2\}$ because of the influence of this parameter on consumption.

The first results obtained through these simulations can be seen in Table 4.2. A delay in the age of retirement clearly has a positive effect on economic growth. Nonetheless, this result needs further research, within this current framework, regarding the simulations introduced in Chapter 3. Furthermore, we find that, given the age of retirement, the greater the life expectancy is, the lower both consumption and physical capital per capita are. This is so because the relationship between workers and retirees decreases. Thus, although savings during the working period increase due to the greater life expectancy after retirement, it does not balance the losses yielded by the greater number of retirees. However, we have shown in Chapter 3 that the existence of both public and private pension systems can contribute to raising individual wealth. This fact is worth considering for future research as well.

Second, we find looking at figures from 4.6 to 4.13 that economic variables do not move in a straight line. In contrast, figures show that economic transitions are cyclical. The importance of this feature is twofold. First, because it rejects the use of linear models for estimating the economic growth of any economy.[15] For example, if we have data of the first hundred years after the initial state, the econometric model will estimate that a funded social security is better than an unfunded social security. Nevertheless, this conclusion does not support the proposition 4.2. Thus, rather than the linear model, our results suggest to use a time series model with sines and cosines. On the other side, the greater the number of cycles, the more stable the economy becomes. Note that to come to this conclusion we have not included from figure 4.7 to figure 4.13 the case in which $\gamma = .75$ and $\tau = 0$. We have done so because in that case the golden rule is an unstable focus. However, the rest of the economic transitions seem to be stable. Concretely, we have found that the economy is more stable under the following conditions: i) the greater the risk aversion is, ii) the higher the age of retirement is, and iii) the greater the payroll tax levied by an unfunded social security is. On the contrary, an economy is more unstable: i) the greater the life expectancy becomes, and ii) when the economy has a funded social security, instead of an unfunded one.

[15] If and only if the economy can reach a golden rule equilibrium.

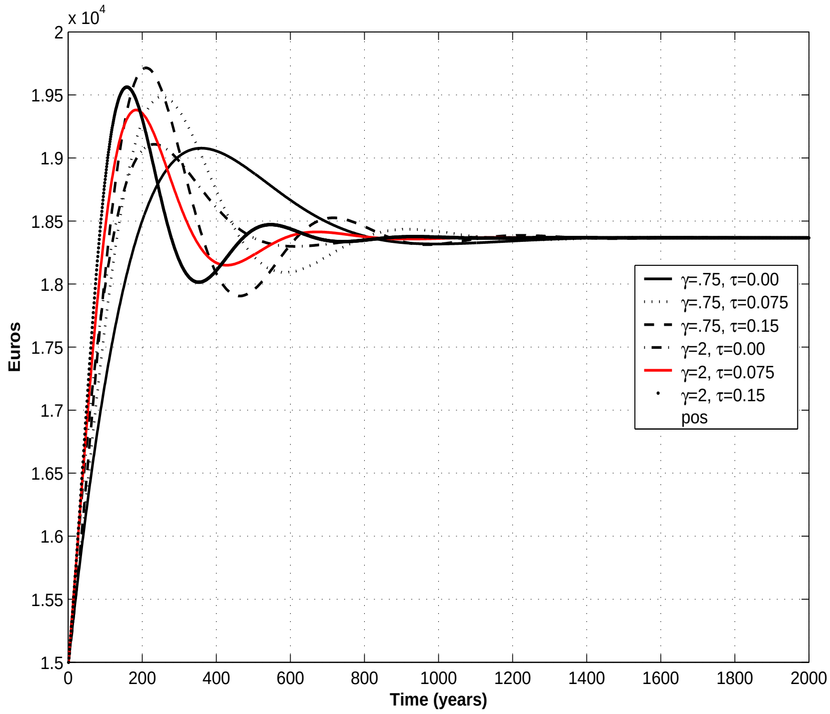
Table 4.1: Calibration

Symbol	Values
δ	.01
n	.0150
T	120
J	{65, 70}
$\Omega(s,x)$	$x = \{1940, 1960, 1980, 2000\}$
A	2093
θ	$\frac{1}{3}$
α	-.131855
$k(0)$	100.000
$c(0)$	15.000

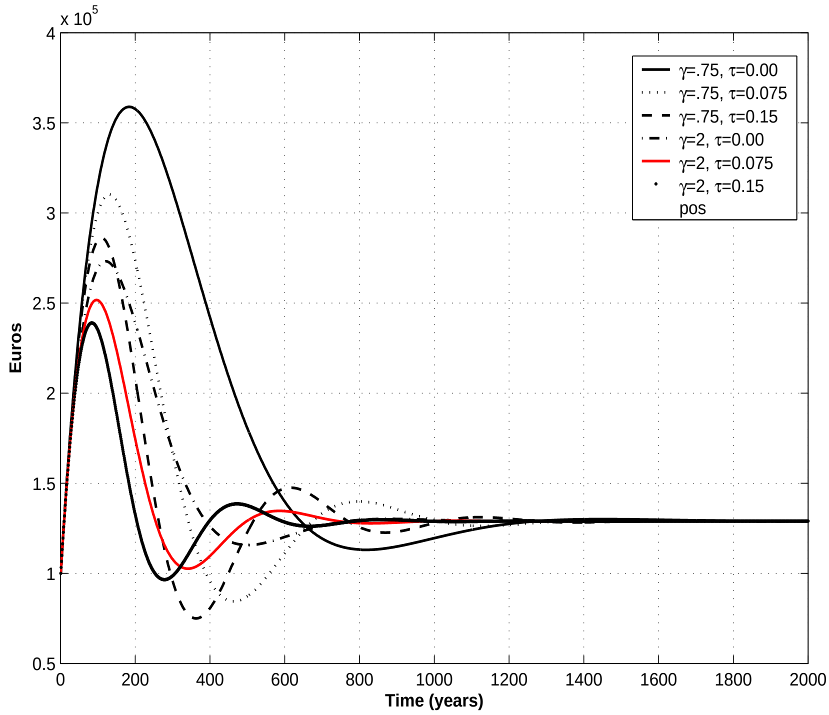
Table 4.2: Steady Consumptions and Physical Capitals per Capita: Golden Rule

$r^* = .0150$	$J = 65$		$J = 70$	
Cohorts	k^*	c^*	k^*	c^*
1940	128.998	18.367	131.536	18.728
1960	125.816	17.914	129.257	18.404
1980	123.855	17.635	127.686	18.180
2000	121.677	17.325	125.864	17.921

Figure 4.6: Economic Transitions from (k(0),c(0)) to the Golden Rule Equilibrium. Case: Cohort 1940, $J = 65$.

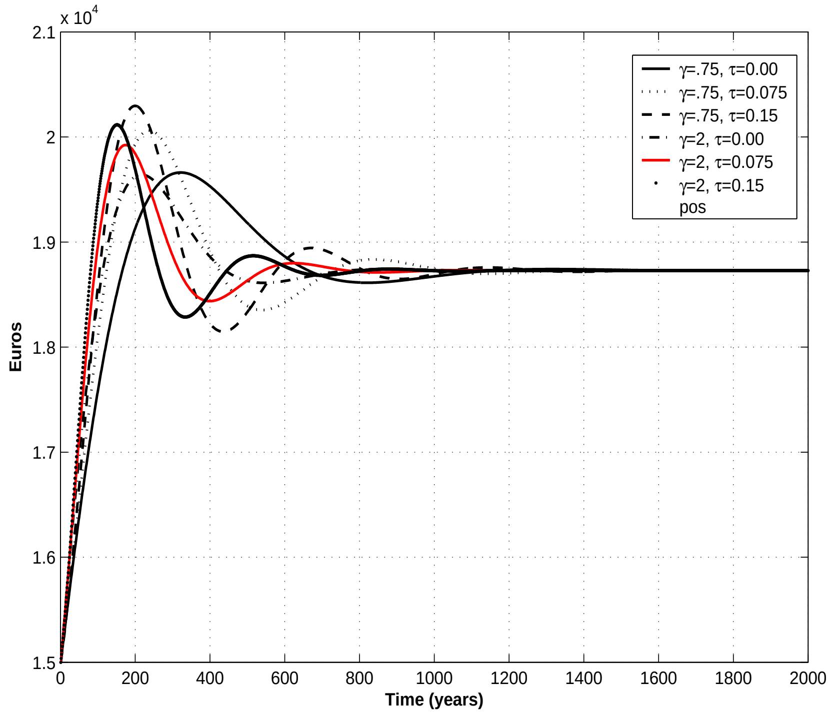


(a) Consumption per capita

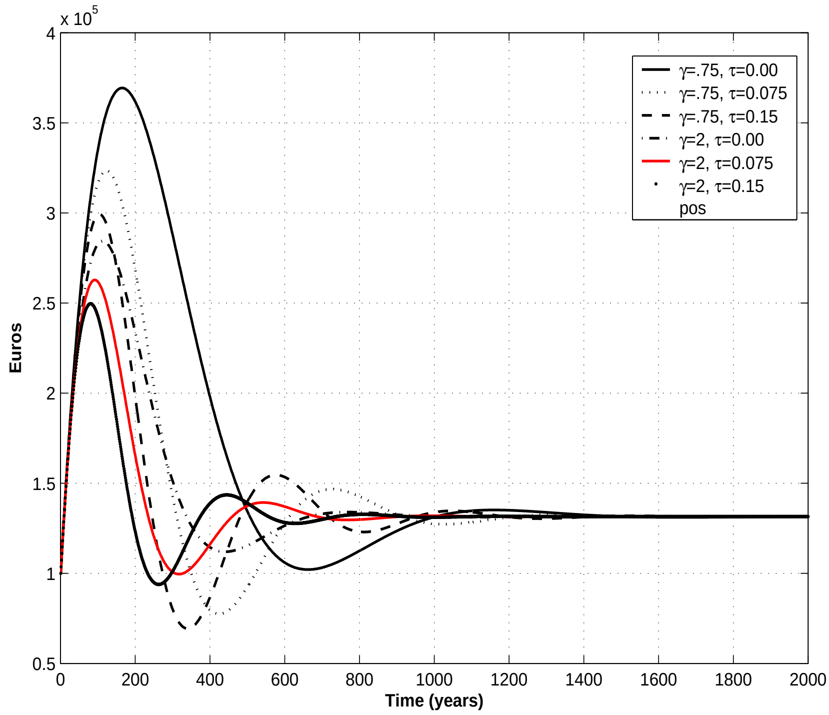


(b) Physical Capital per capita

Figure 4.7: Economic Transitions from (κ(0),c(0)) to the Golden Rule Equilibrium. Case: Cohort 1940, $J = 70$.

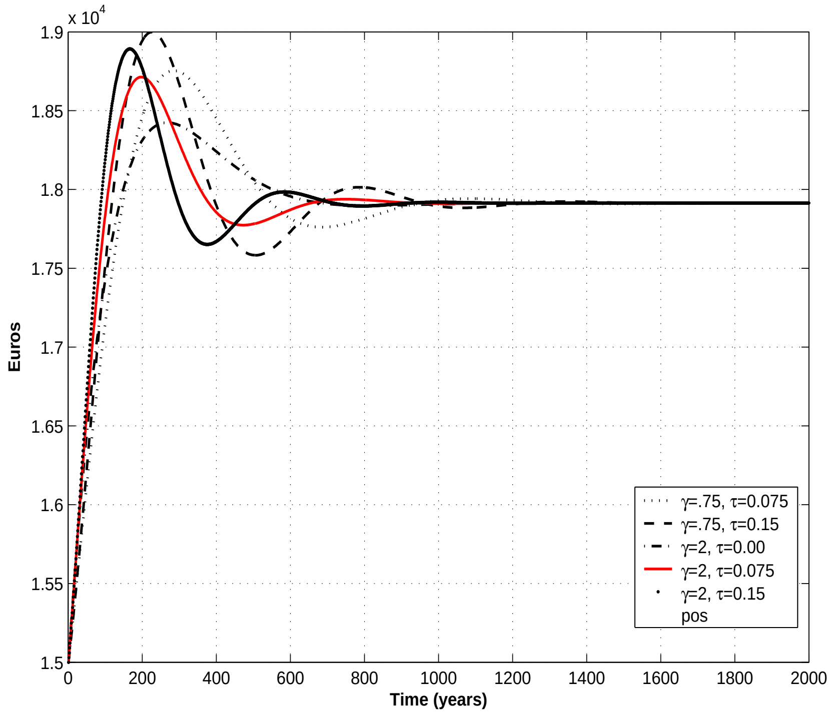


(a) Consumption per capita

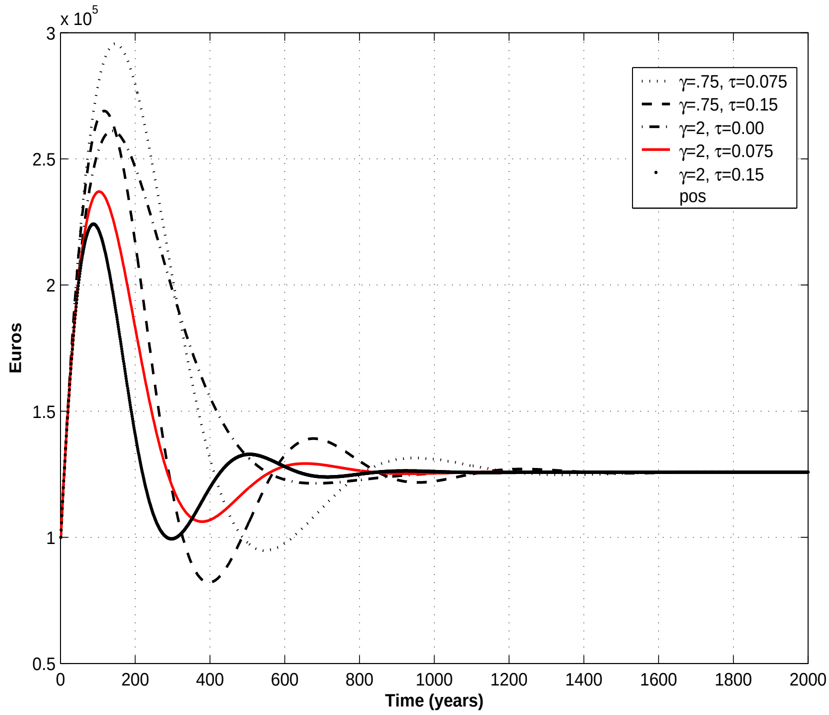


(b) Physical Capital per capita

Figure 4.8: ECONOMIC TRANSITIONS FROM (κ(0),c(0)) TO THE GOLDEN RULE EQUILIBRIUM. CASE: COHORT 1960, $J = 65$.

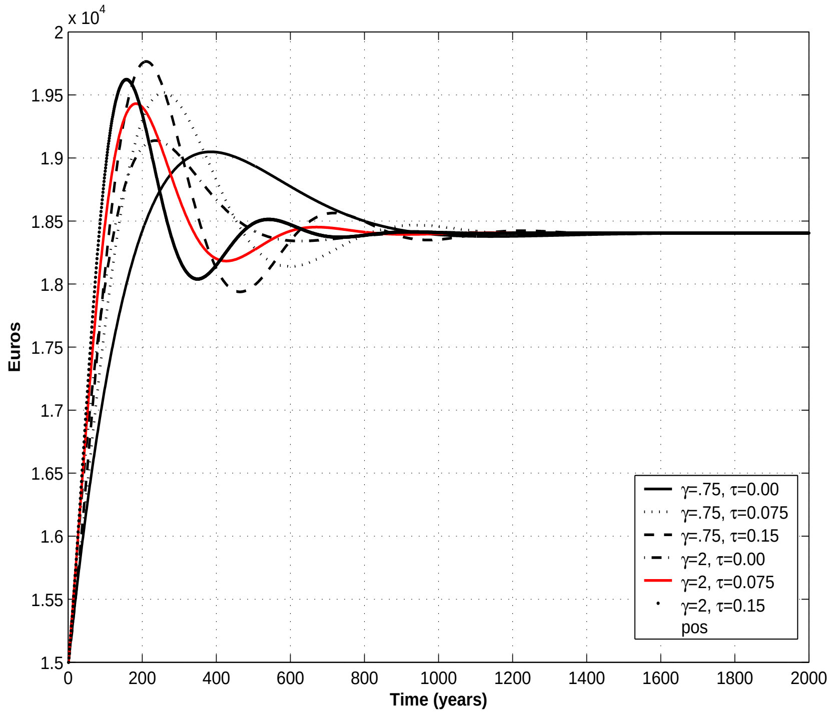


(a) Consumption per capita

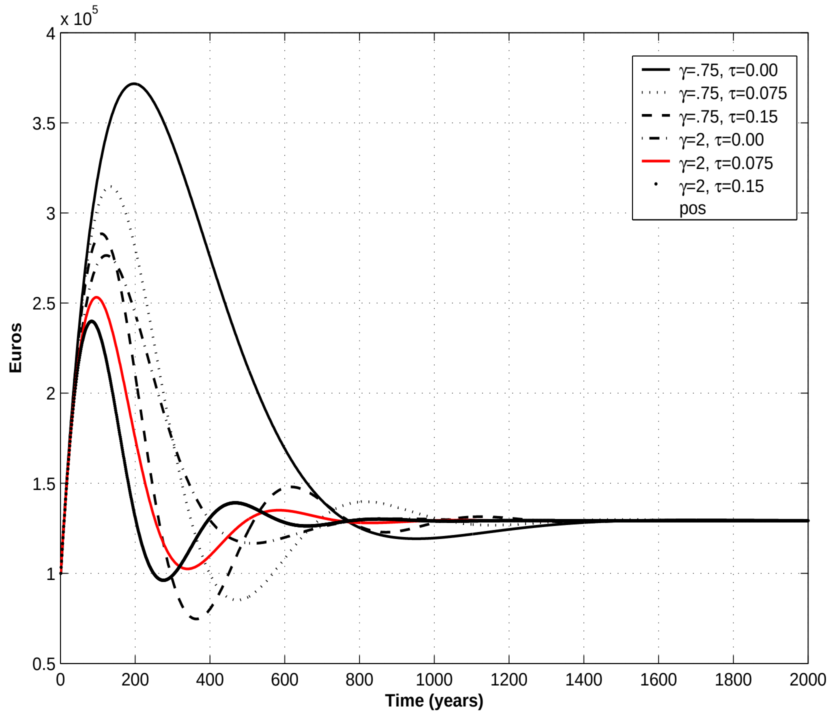


(b) Physical Capital per capita

Figure 4.9: Economic Transitions from (κ(0),c(0)) to the Golden Rule Equilibrium. Case: Cohort 1960, $J = 70$.

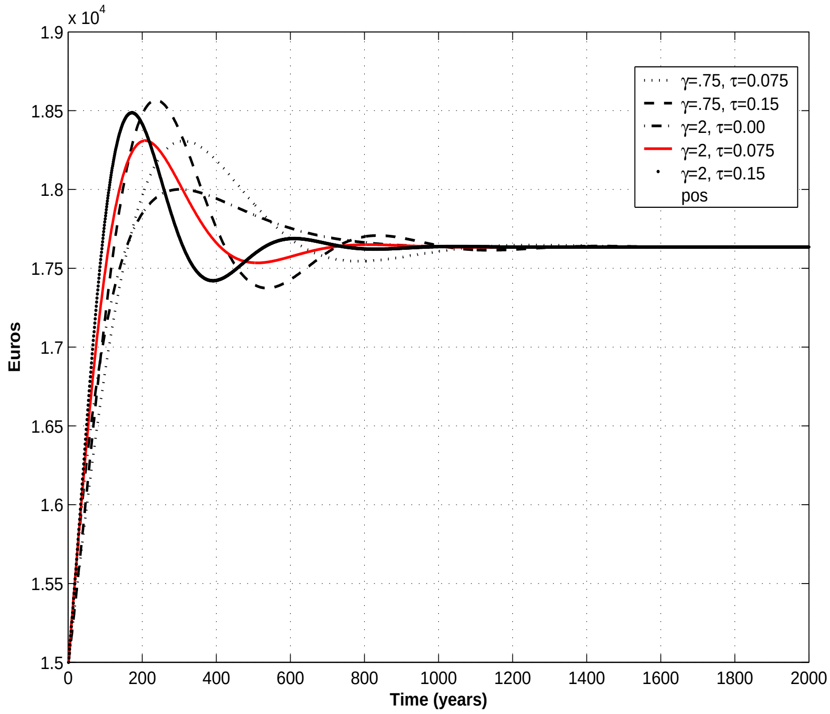


(a) Consumption per capita

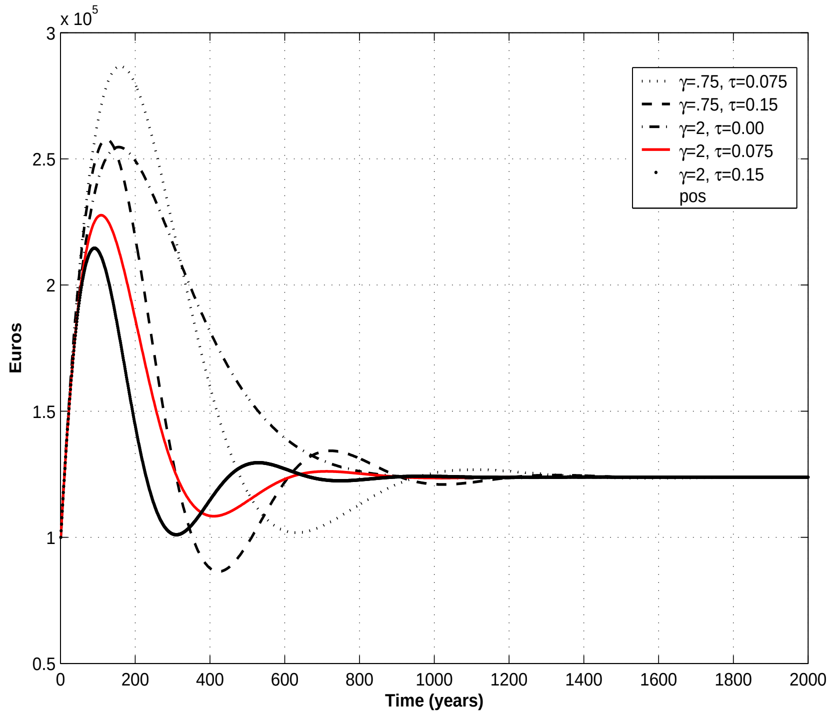


(b) Physical Capital per capita

Figure 4.10: Economic Transitions from (k(0),c(0)) to the Golden Rule Equilibrium. Case: Cohort 1980, $J = 65$.

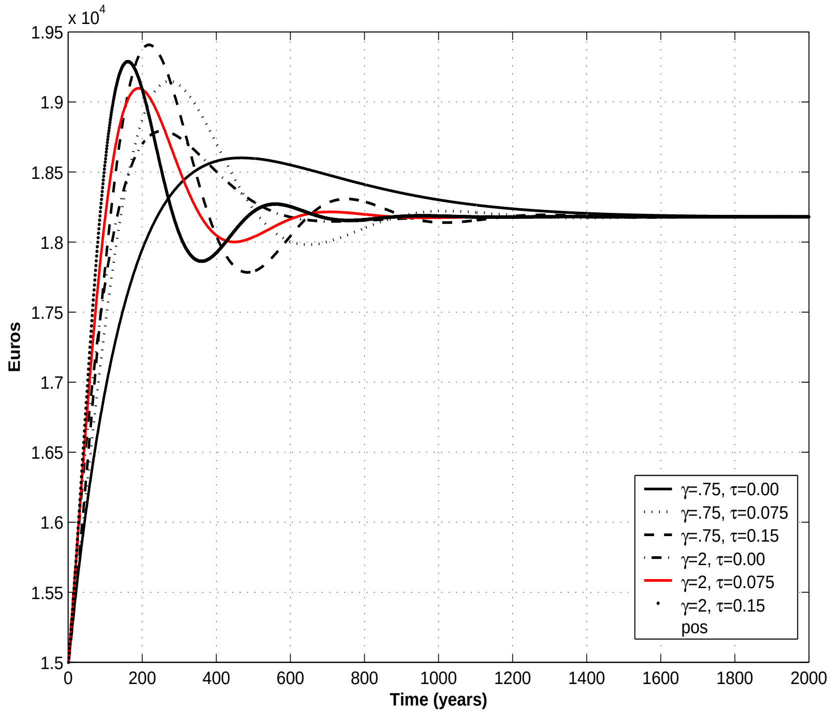


(a) Consumption per capita

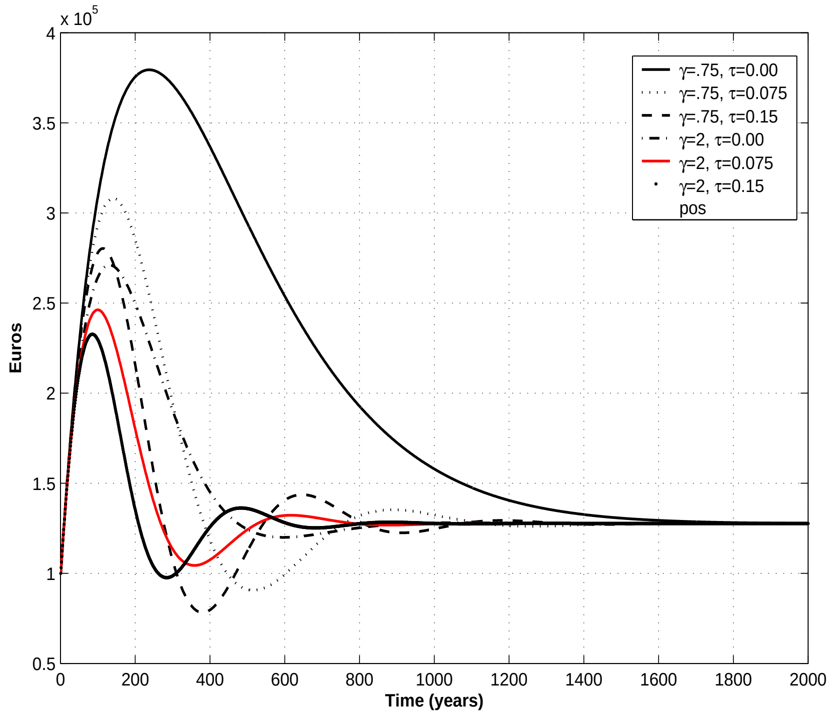


(b) Physical Capital per capita

Figure 4.11: Economic Transitions from (k(0),c(0)) to the Golden Rule Equilibrium. Case: Cohort 1980, $J = 70$.

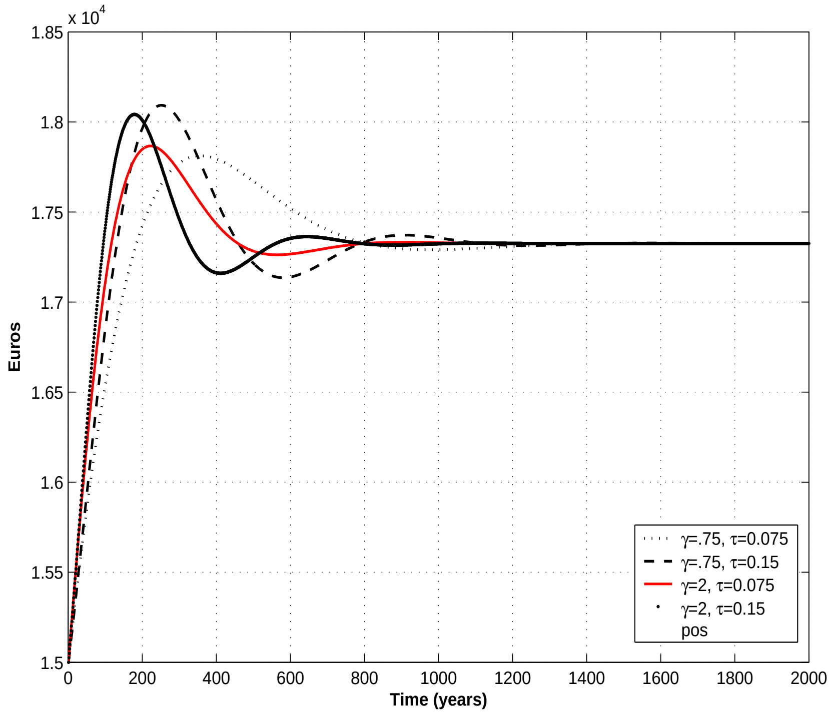


(a) Consumption per capita

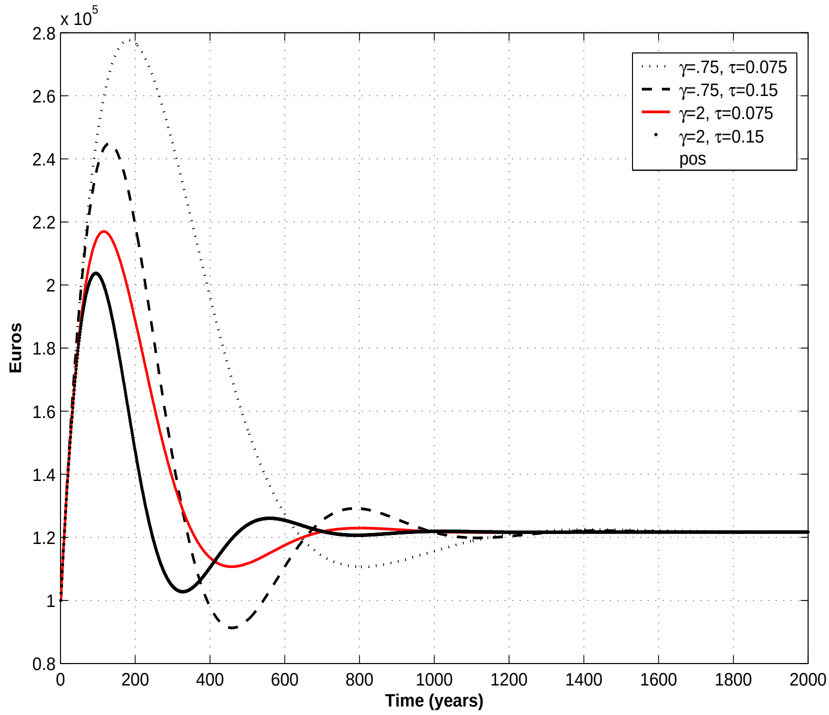


(b) Physical Capital per capita

Figure 4.12: Economic Transitions from (k(0),c(0)) to the Golden Rule Equilibrium. Case: Cohort 2000, $J = 65$.

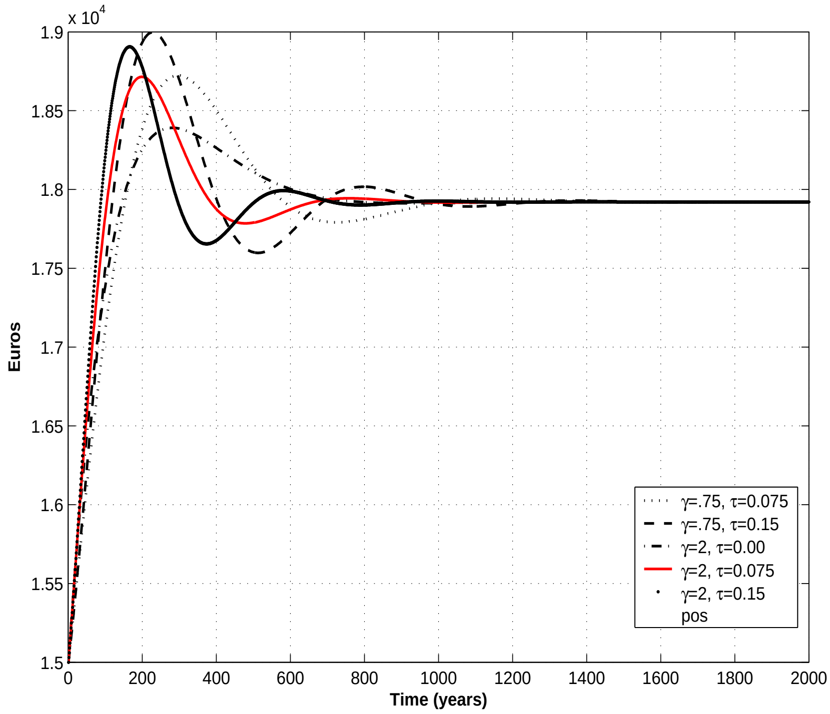


(a) Consumption per capita

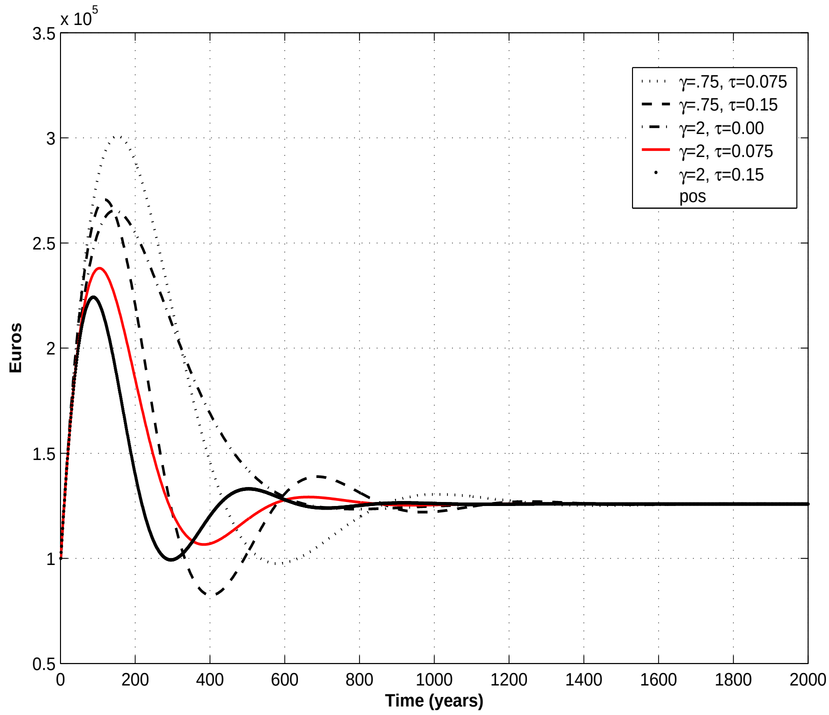


(b) Physical Capital per capita

Figure 4.13: ECONOMIC TRANSITIONS FROM (K(0),C(0)) TO THE GOLDEN RULE EQUILIBRIUM. CASE: COHORT 2000, $J = 70$.



(a) Consumption per capita



(b) Physical Capital per capita

Chapter 5

Conclusiones y Futuras Líneas de Investigación

El objetivo que hemos perseguido en la presente tesis ha sido ilustrar y explicar la conveniencia de un sistema público de pensiones. Los resultados obtenidos, tanto teóricos como simulados, sugieren que la escasa demanda de seguros de vida y pensiones podría explicarse por medio del comportamiento miope de los individuos. Este comportamiento justificaría la necesidad de un sistema público de pensiones. Por otra parte, cuando el agente económico es miope y egoista y las instituciones financieras no permiten que las personas mueran endeudadas, una seguridad social de capitalización podría contribuir a aumentar la riqueza de los agentes económicos antes de su jubilación. Así, la carga fiscal derivada del envejecimiento poblacional podría ser mitigada mediante la aplicación de este sistema. Por último, también demostramos, siempre que la economía crezca según la regla de oro, que una seguridad social de reparto es preferible frente a una seguridad social de capitalización.

En el Capítulo 2 hemos demostrado que las rentas actuariales de carácter privado no siempre suponen, para el agente económico, una ganancia de bienestar frente a las rentas financieras. En particular, la decisión de contratar bien una renta actuarial, o bien una renta financiera, depende de la relación entre el valor presente de los ingresos futuros y el nivel inicial de riqueza. Este resultado se explica bajo la hipótesis de que el agente económico es egoista y no le preocupa el hecho de tener una posición deudora no cubierta por un seguro. Por lo tanto, los agentes económicos pueden decidir el no contratar una renta actuarial y, como consecuencia, malgastar sus recursos financieros antes de fallecer. Ahora bien, a pesar de que la Seguridad Social puede ayudar al agente a no malgastar sus recursos, una contribución a la Seguridad Social que no sea óptima podría producir, en el corto plazo, un efecto expulsión sobre el stock de capital. Al respecto, en esta tesis demostramos, a medida que la contribución a la Seguridad Social es mayor, que los individuos egoistas están más predispuestos a invertir su riqueza en bonos que en seguros. Sin embargo, también demostramos que, aunque la Seguridad Social desincentiva la compra de rentas actuariales y pensiones privadas, el sistema público puede ayudar a aumentar la riqueza. Por consiguiente, podemos afirmar que existe un tipo de contribución a la Seguridad Social que maximiza la suma del capital invertido en el sistema público y privado de pensiones. En particular, hemos obtenido que un tipo de contribución "óptimo", en torno al 6 por ciento, tiene como efecto negativo que el agente económico no invierta en seguros al comienzo de su ciclo vital, pero como efecto positivo que su riqueza a la edad de jubilación sea un 17 por ciento mayor.

El modelo económico de equilibrio parcial, utilizado para explicar los anteriores resultados, no permite sin embargo estudiar las consecuencias derivadas en el largo plazo del posible efecto expulsión del capital. Tal es así que este modelo no tiene en cuenta los cambios que producen la estructura poblacional y el consumo agregado en el precio futuro de los factores productivos. Por lo cual, con el objetivo de analizar los efectos económicos producidos por la Seguridad Social en el largo plazo, en el Capítulo 4 se desarrolla un modelo de crecimiento económico con generaciones solapadas, cuyo marco contable es longitudinal en vez de transversal. Gracias a este nuevo marco contable, hemos demostrado que, en una economía compuesta por individuos egoistas y racionales y por compañías aseguradoras que ofrecen seguros actuarialmente justos, los sistemas de capitalización y de reparto alcanzan los estados estacionarios correspondientes a la regla de oro y a la regla de oro modificada. Por tanto, la seguridad social de reparto no genera un efecto expulsión sobre el stock de capital físico. Además, cuando la economía crece al mismo ritmo que la población (regla de oro), ambos sistemas de financiación de la seguridad social proporcionan el mismo bienestar. Sin embargo, el sistema de reparto confiere una mayor estabilidad a la economía. Por todo ello, parece razonable afirmar que cuando la economía satisface la regla de oro, una seguridad social de reparto es preferible frente a una seguridad social de capitalización.[1]

En resumen, el cuerpo de esta tesis compuesto por los capítulos del 2 al 4, desde el modelo de equilibrio parcial al modelo de crecimiento económico con generaciones solapadas, extiende el marco teórico para analizar los efectos de la Seguridad Social en el crecimiento económico. De hecho, cada capítulo aporta nuevos resultados, tanto teóricos como empíricos, con respecto a los sistemas de reparto y capitalización de la seguridad social. No obstante, estos resultados no agotan las líneas de investigación futuras, pues existen múltiples formas de aplicación de la presente tesis con datos económicos reales.

Una primera línea de investigación es incorporar el comportamiento miope, descrito en los capítulos 2 y 3, en el modelo de crecimiento económico con generaciones solapadas. La necesidad de encontrar métodos alternativos de financiación a la Seguridad Social justifica esta línea de investigación. Así, en esta tesis hemos demostrado que, cuando los agentes económicos aseguran sus ahorros, una seguridad social de reparto es más conveniente que una seguridad social de capitalización. Sin embargo, ese escenario no se da en la realidad. Por lo tanto, el incorporar la posibilidad de que los individuos no aseguren su riqueza, parece ser lo más acertado. De este modo, la búsqueda de un tipo de contribución a la Seguridad Social que maximize el stock de capital nacional podrá ser encontrado, no sólo para el corto plazo, sino también para el largo plazo.

Una segunda línea de investigación es analizar el modelo de crecimiento económico con generaciones solapadas, incorporando tanto las transiciones demográficas como las expectativas racionales. En este caso, las herramientas econométricas (por ejem-

[1] El resto de estados estacionarios no han sido simulados en esta tesis pues o bien son inestables, o bien no son Pareto-eficientes.

plo, series temporales y cadenas de markov) constituirán el marco analítico fundamental. En definitiva, esta línea de investigación será aplicada al estudio de la viabilidad de cualquier sistema de Seguridad Social.

Bibliography

Andrew Abel. "Precautionary Saving and Accidental Bequest". *The American Economic Review*, 75(4):777–791, September 1985.

Andrew B. Abel. "Capital Accumulation and Uncertain Lifetimes with Adverse Selection". *Econometrica*, 54(5):1079–1098, September 1986.

A. Ando and F. Modigliani. "Test of the Life-Cycle Hypothesis of Savings". *Bulletin of the Oxford Institute of Economist and Statistics*, 19:99–124, 1957.

Alan Auerbach. *Work, Health and Income among the Elderly*, chapter Life Insurance of the Elderly: Adequacy and Determinants, pages 229–268. The Brooking Institution, 1987. Reprinted by permission in Kotlikoff, L. (2001) *Essays on Saving, Bequests, Altruism, and Life-Cycle Planning*. Massachussetts Institute of Technology, 387-425.

Alan J. Auerbach and Laurence J. Kotlikoff. *Dynamic Fiscal Policy*. Cambridge University Press, 1987.

Robert Barro and James Friedman. "On Uncertain Lifetimes". *The Journal of Political Economy*, 85(4):843–849, August 1977.

Douglas Bernheim. "How Strong Are Bequest Motives? Evidence Based on Estimates of the Demand for Life Insurance and Annuities". *The Journal of Political Economy*, 99(5):899–926, 1991.

Olivier J. Blanchard. "Debt, Deficits, and Finite Horizons". *The Journal of Political Economy*, 93(2):223–247, April 1985.

Antoine Bommier. “Aversion for Early Death and the Structure of Time Preference”. Working Paper LEA 01-06, September 2001.

Antoine Bommier. “Risk Aversion, Intertemporal Elasticity of Substitution and Correlation Aversion”. Working Paper LEA 03-07, 2003a.

Antoine Bommier. “Uncertain Lifetime and Intertemporal Choice”. Working Paper LEA 01-08, 2003b.

Antoine Bommier and Ronald D. Lee. “Overlapping Generations Models with Realistic Demography”. *Journal of Population Economics*, 16:135–160, 2003.

M. Boskin and M. Hurd. “The Effect of Social Security on Early Retirement”. *Journal of Public Economics*, 10(3):361–377, December 1978.

Jeffrey Brown and Mark Warshawsky. “Longevity-Insured Retirement Distributions from Pension Plans: Market and Regulatory Issues”. NBER Working Paper No. 8064, 2001.

Jeffrey R. Brown. “Redistribution and Insurance: Mandatory Annuitization with Mortality Heterogeneity”. *The Journal of Risk and Insurance*, 70(1):17–41, 2003.

Guillermo A. Calvo and Maurice Obstfeld. “Optimal Time-Consistent Fiscal Policy with Finite Lifetimes”. *Econometrica*, 56(2):411–432, March 1988.

David Cass. “Optimum Growth in an Aggregate Model of Capital Accumulation”. *Review of Economic Studies*, 32:233–240, July 1965.

J. C. Conesa and D. Krueger. “Social Security Reform with Heterogeneous Agents”. *Review of Economic Dynamics*, 2:757–795, 1999.

Thomas Davidoff, Jeffrey Brown, and Peter Diamond. “Annuities and Individual Welfare”. *The American Economic Review*, 95(5):1573–1590, December 2005.

James Davies. “Uncertain Lifetime, Consumption, and Dissaving in Retirement”. *The Journal of Political Economy*, 89(3):561–577, June 1981.

Zvi Eckstein, Martin S. Eichenbaum, and Dan Peled. "The Distribution of Wealth and Welfare in the Presence of Incomplete Annuity Markets". *The Quarterly Journal of Economics*, 100(3):789–806, August 1985.

Martin Feldstein. "Social Security, Induced Retirement, and Aggregate Capital Accumulation". *Journal of Political Economy*, 82(5):905–926, 1974.

Irving Fisher. *The Theory of Interest*. Pickering and Chatto, London, 1977.

Benjamin Friedman and Mark Warshawsky. "The Cost of Annuities: Implication for Saving Behavior and Bequests". *The Quarterly Journal of Economics*, 105(1): 135–154, February 1990.

Luisa Fuster. "Is Altruism Important for Understanding the Long-Run Effects of Social Security?". *Review of Economic Dynamics*, 2:616–637, 1999.

Luisa Fuster, Ayşe İmrohoroğlu, and Selahattin İmrohoroğlu. "A Welfare Analysis of Social Security in a Dynastic Framework". *International Economic Review*, 44 (4):1247–1274, November 2003.

David Gale. "Pure Exchange Equilibrium of Dynamic Economic Models". *Journal of Economic Theory*, 6:12–36, 1973.

Li Gan, Michael Hurd, and Daniel McFadden. "Individual Subjective Survival Curve". NBER Working Paper No. 9480, January 2003.

Jagadeesh Gokhale, Laurance Kotlikoff, and Mark Warshawsky. "Life-Cycle Savings, Limits on Contributions to DC Pension Plans, and Lifetime Tax Benefits". NBER Working Paper No. 8170, March 2001a.

Jagadeesh Gokhale, James Sefton, and Martin Wale. "Simulating the Transmission of Wealth Inequality via Bequests". *Journal of Public Economics*, 79(1):98–128, January 2001b.

Nils Hakansson. "Optimal Investment and Consumption Strategies Under Risk, an Uncertain Lifetime, and Insurance". *International Economic Review*, 10(3): 443–466, October 1969.

Daniel Hamermesh. "Expectations, Life Expectancy, and Economic Behavior". *The Quarterly Journal of Economics*, 100(2):389–408, 1985.

R. Glenn Hubbard. *Issues in Pension Economics*, chapter Uncertain Lifetime, Pensions, and Individual Saving, pages 175–206. NBER, 1987.

Michael Hurd. "Mortality Risk and Bequest". *Econometrica*, 57(4):779–813, July 1989.

Michael Hurd and Kathleen McGarry. "Evaluation of the Subjective Probabilities in the Health and Retirement Study". *The Journal of Human Resources*, 30(4): S268–S292, 1995.

Michael Hurd and Kathleen McGarry. "The Predictive Validity of the Subjective Probabilities of Survival". *The Economic Journal*, 112:966–985, October 2002.

A. İmrohoroğlu, S. İmrohoroğlu, and D. Joines. "Social Security in an Overlapping Generations Economy with Land". *Review of Economic Dynamics*, 2:638–665, 1999.

Tjalling C. Koopmans. *The Economic Approach to Development Planning*, chapter "On the Concept of Optimal Economic Growth". North-Holland, Amsterdam, 1965.

Laurance Kotlikoff and Lawrence Summer. "The Role of Intergenerational Transfers in Aggregate Capital Accumulation". *The Journal of Political Economy*, 89(2): 706–732, August 1981.

Laurence Kotlikoff and Avia Spivak. "The Family as an Incomplete Annuities Market". *The Journal of Political Economy*, 89(2):372–391, April 1981.

Laurence J. Kotlikoff, Stephen Johnson, and William Samuelson. *Essays on Saving, Bequest, Altruism, and Life-Cycle Planning*, chapter Can People Compute? An

Experimental Test of the Life-Cycle Consumption Model, pages 335–386. Massachusetts Institute of Technology, 2001.

David Levhari and Leonard Mirman. "Savings and Consumption with an Uncertain Horizon". *The Journal of Political Economy*, 85(2):265–281, April 1977.

Harry Markowitz. "Portfolio Selection". *The Journal of Finance*, 7(1):77–91, March 1952.

Robert C. Merton. "Optimum Consumption and Portfolio Rules in a Continuous-Time Model". *Journal of Economic Theory*, 3:373–413, December 1971.

Robert C. Merton. *Financial Aspects of the Unitated States Pension System*, chapter On the Role of Social Security as a Means for Efficient Risk Sharing in an Economy Where Human Capital Is Not Tradable, pages 325–358. Project Report. National Bureau of Economic Research, 1983.

Olivia Mitchell, James Poterba, Mark Warshawsky, and Jeffrey Brown. "New Evidence on the Money's Worth of Individual Annuities". *The American Economic Review*, 89(5):1299–1318, December 1999.

F. Modigliani and R. Brumberg. *Post Keynesian Economics*, chapter Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data, pages 388–436. 1954.

Edmund S. Phelps. *Golden Rules of Economic Growth*. Norton, New York, 1966.

F.P. Ramsey. "A Mathematical Theory of Saving". *The Economic Journal*, 38(152): 543–559, December 1928.

Scott F. Richard. "Optimal Consumption, Porfolio and Life Insurance Rules for an Uncertain Lived Individual in a Continuous Time Model". *Journal of Financial Economics*, 2:187–203, 1975.

Miguel Sanchez-Romero. "Welfare Gain and the Demand for Annuities". Economic Analysis Working Paper Series 2/2005, November 2005.

William F. Sharpe. “Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk”. *The Journal of Finance*, 19(3):425–442, September 1964.

Robert M. Solow. “A Contribution to the Theory of Economic Growth”. *The Quarterly Journal of Economics*, 70(1):65–94, Febreary 1956.

S. C. Tsiang. “The Rationale of the Mean-Standard Deviation Analysis, Skewness Preference, and the Demand for Money”. *The American Economic Review*, 62(3): 354–371, June 1972.

Mark Warshawsky. “Private Annuity Market in the United States: 1919-1984”. *The Journal of Risk and Insurance*, 55:518–528, September 1988.

Menahem Yaari. “Uncertain Lifetime, Life Insurance, and the Theory of the Consumer”. *The Review of the Economic Studies*, 5(3):304–317, September 1965.

Tadashi Yagi and Yasuyuki Nishigaki. “The Inefficiency of Private Constant Annuities”. *The Journal of Risk and Insurance*, 60(3):385–412, 1993.